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ACTIVE POWER MINIMIZATION AND POWER ABSORPTION IN A PLATE WITH FORCE AND MOMENT EXCITATION

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This paper presents a comparison of the active control strategies of minimizing the total power supplied to a plate and maximizing the power absorbed by the secondary source. Force and moment excitations of infinite and finite plates are considered. For an infinite plate analytic solutions can be obtained for the total power supplied to the plate by the primary and secondary actuator arrays when using the two control strategies. Minimizing the power supplied by a primary force or moment with a secondary force or moment can produce large attenuation provided that the two sources are close together compared with a flexural wavelength. Minimizing the power supplied by a primary force with a secondary moment can also give attenuation of up to 5 dB when the spacing between the sources is about 0·3 times the flexural wavelength. In contrast to the acoustic case, for the infinite plate the total power supplied is generally reduced when the power absorbed by the secondary source is maximized. On a finite panel, however, maximizing the power absorbed by the secondary source can significantly increase the total power supplied. The strategy of minimizing the total power supplied can give considerably larger values of attenuation for finite rather than infinite plates, particularly at higher frequencies. This is because the structural modes can be actively controlled. A particularly efficient secondary actuator for total power minimisation appears to be an independently adjustable, but collocated, force and moment pair.

1. INTRODUCTION

In the past three decades, researchers have started to describe the vibrations of flexible structures by using power. Noiseaux [1], Pavic [2], Verheij [3] and Williams [4] provided some preliminary studies on the measurement of structural power, while Goyder and White [5–7] presented a detailed and systematic theoretical study of the structural power in one- and two-dimensional flexible structures. It has been found that structural power is a single parameter that can efficiently represent the main effects of the vibrations of a structure and is particularly useful when multi-degree-of freedom vibrations occur. Power has also been used to describe the action of various systems for the active control of vibration. Miller *et al*. [8], Pan and Hansen [9], Schwenk *et al*. [10], Redman-White *et al*. [11], Gibbs and Fuller [12], and Parakah-Asante and Craig [13] have investigated the possibility of actively controlling the vibrations of a one-dimensional system by using power as the minimized cost function. Koh and White [14], Nam *et al*. [15] and Pavic [16] have considered the possibility of controlling vibrations of two-dimensional structures by minimizing the power input to them. More recently, researchers have studied more

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complicated systems composed of several members; Pan *et al*. [17] and Gardonio *et al*. [18, 19] have investigated the possibility and the advantages of using power as the cost function in isolator systems with active mounts.

This paper is concerned with the physical limitations of using different strategies for active vibration control on plates. It is assumed that an array of primary forces and moments acting on a plate are to be controlled by another array of forces and moments acting as secondary sources. In order to understand the complicated behaviour of such multi-channel systems, it is important to have gained insight from simpler systems with a limited number of primary and secondary inputs. A general framework is presented for the calculation of the effect of minimizing the total power into the plate and analytic results are obtained for some simple cases. Jenkins *et al*. [20] considered the important case of minimizing the total input power supplied to an infinite plate by a point force primary source and a point force secondary source. Total power minimization is also considered in this paper for systems in which the primary and/or the secondary inputs are moments. Whereas for point force inputs very large attenuation can be achieved when the two forces are close together compared with a flexural wavelength on the plate, controlling a primary point force with a secondary moment is found to be most successful if the moment is placed about a third of a flexural wavelength from the point force, in which case the total power input can be reduced by about 5 dB.

Another strategy for active control is to maximize the power absorption of the secondary source. For acoustic monopole sources, Elliott *et al*. [21] showed that this strategy produces disastrous increases in total power output if the separation between the primary and secondary source is small compared with the acoustic wavelength. Rather surprisingly, this effect does not occur when the power absorption of a secondary force close to a primary force is maximized on an infinite plate. The total power input is reduced by 6 dB as the separation between the forces becomes small. This is thought to be due to the absence of the strong near-field component in quadrature with the source strength, which is present in the acoustic case.

Finally, the important practical case of a finite plate is considered, the response of which is modelled by using a modal summation. The effect of power minimisation is generally to suppress the modal resonances which the secondary force or moment can efficiently couple into. Significant control can now be obtained at resonance for source separation distances which are large compared with the flexural wavelength, provided that the secondary source can effectively couple into the mode. The control strategy of maximizing power absorbed by the secondary sources can, however, produce large increases in total power input to the finite plate; which is consistent with the behaviour in beams [22] and observed in finite acoustic systems using this control strategy [21].

2. MINIMIZING TOTAL POWER OUTPUT

2.1. A SET OF PRIMARY FORCES CONTROLLED BY A SET OF SECONDARY FORCES

In this section the case is considered of an infinite plate excited by a set of primary forces and controlled by a set of secondary forces adjusted by using the strategy of total power minimization.

For single frequency excitation a force vector can be defined which is made up of a set of N complex forces applied on the plate $\mathbf{f}^{\mathrm{T}} = \{f_1, f_2, \ldots, f_N\}$ which are proportional to $\exp(\mathrm{j}\omega t)$. In a similar manner, one can define a velocity vector made up of a set of velocities at the points of application of the forces as $\mathbf{v}^{\mathrm{T}} = \{v_1, v_2, \ldots, v_N\}$. Thus, the total power output of the force array into the plate can be expressed as

$$W_T = \tfrac{1}{2}\,\mathrm{Re}\{\mathbf{f}^{\mathrm{H}}\mathbf{v}\}, \tag{1}$$

where Re{} denotes the real part of a complex number and the superscript H denotes the Hermitian, complex conjugate, transpose. Furthermore, it is possible, by using the mobility matrix $\mathbf{M} = [M_{ij}]$, to express $\mathbf{v}$ as

$$\mathbf{v} = \mathbf{M}\mathbf{f} \tag{2}$$

so that W_T can be written as [5]

$$W_T = \tfrac{1}{2}\,\mathrm{Re}\,\{\mathbf{f}^{\mathrm{H}}\mathbf{M}\mathbf{f}\}, \tag{3}$$

or, equivalently [5],

$$W_T = \tfrac{1}{2}\,\mathrm{Re}\,\{\mathbf{v}^{\mathrm{H}}\mathbf{M}^{-1}\mathbf{v}\} = \tfrac{1}{2}\,\mathrm{Re}\,\{\mathbf{v}^{\mathrm{H}}\mathbf{Z}\mathbf{v}\}, \tag{4}$$

where $\mathbf{Z}$ is the impedance matrix defined by $\mathbf{Z} = \mathbf{M}^{-1}$. In general, however, it is the forces that can be controlled, and so equation (3) is most applicable here. Minimizing the expression of the total power output, W_T, leads to the optimal value of the secondary source, $\mathbf{f}_{Sopt}$ [20]. Assuming reciprocity, one can show that $\mathbf{M}$ is symmetric. In this case equation (3) can be written as

$$W_T = \tfrac{1}{2}\,\mathbf{f}^{\mathrm{H}}\mathrm{Re}\,\{\mathbf{M}\}\mathbf{f}. \tag{5}$$

Now, if one writes $\mathbf{M} = \mathbf{R} + \mathrm{j}\mathbf{X}$, where $\mathbf{R}$ and $\mathbf{X}$ are the real and imaginary parts of $\mathbf{M}$, then equation (5) becomes

$$W_T = \tfrac{1}{2}\,\mathbf{f}^{\mathrm{H}}\mathbf{R}\mathbf{f}. \tag{6}$$

If one considers the total force vector as having components due to the primary and secondary sources, $\mathbf{f}^{\mathrm{T}} = \{\mathbf{f}_P^{\mathrm{T}}\ \ \mathbf{f}_S^{\mathrm{T}}\}$, it is possible to decompose $\mathbf{R}$ as

$$\mathbf{R} = \begin{bmatrix} \mathbf{R}_{PP} & \mathbf{R}_{PS} \\ \mathbf{R}_{SP} & \mathbf{R}_{SS} \end{bmatrix}, \tag{7}$$

where $\mathbf{R}_{PP}$ contains all terms due to the primary force distribution, and $\mathbf{R}_{SS}$ contains all terms due to the secondary force distribution, both of which are symmetric. $\mathbf{R}_{PS}$ and $\mathbf{R}_{SP}$ represent interference terms between the two distributions and $\mathbf{R}_{SP} = \mathbf{R}_{PS}^{\mathrm{T}}$ due to reciprocity. By using equation (6), the total power output can now be expressed by [20, 23, 24]

$$W_T = \tfrac{1}{2}\,[\mathbf{f}_P^{\mathrm{H}}\,\mathbf{R}_{PP}\,\mathbf{f}_P + \mathbf{f}_P^{\mathrm{H}}\,\mathbf{R}_{PS}\,\mathbf{f}_S + \mathbf{f}_S^{\mathrm{H}}\,\mathbf{R}_{SP}\,\mathbf{f}_P + \mathbf{f}_S^{\mathrm{H}}\,\mathbf{R}_{SS}\,\mathbf{f}_S], \tag{8}$$

which is a quadratic function of the real and imaginary parts of the elements of $\mathbf{f}_S$, and minimizing this quadratic function one finds the optimal secondary force [20, 23, 24] to be

$$\mathbf{f}_S = \mathbf{f}_{Sopt} = -\mathbf{R}_{SS}^{-1}\,\mathbf{R}_{SP}\,\mathbf{f}_P. \tag{9}$$

The minimum total power output is then

$$W_{T\,min} = \tfrac{1}{2}\,[\mathbf{f}_P^{\mathrm{H}}\,(\mathbf{R}_{PP} - \mathbf{R}_{PS}\,\mathbf{R}_{SS}^{-1}\,\mathbf{R}_{SP})\mathbf{f}_P]. \tag{10}$$

The total power output of primary distribution without control is

$$W_{PP} = \tfrac{1}{2}\,[\mathbf{f}_P^{\mathrm{H}}\,\mathbf{R}_{PP}\,\mathbf{f}_P], \tag{11}$$

so equation (8) can be written in terms of power reduction, which is the ratio of total power with and without optimal secondary distribution:

$$W_{T\,min}/W_{PP} = 1 - \mathbf{f}_p^{\mathrm{H}}\,\mathbf{R}_{PS}\,\mathbf{R}_{SS}^{-1}\,\mathbf{R}_{SP}\,\mathbf{f}_P/\mathbf{f}_P^{\mathrm{H}}\,\mathbf{R}_{PP}\,\mathbf{f}_P. \tag{12}$$

For the particular case of a single primary force controlled by a single secondary force, a distance r away, one has $\mathbf{f}_P = f_P$, $\mathbf{f}_S = f_S$, $\mathbf{R}_{PS} = \mathbf{R}_{SP} = \beta_0\,\mathrm{J}_0(kr)$ [20], where k is the wave number, $\beta_0 = \omega/8Bk^2$ and B is the bending stiffness of the plate, and $\mathbf{R}_{PP} = \mathbf{R}_{SS} = \beta_0$. Equation (9) then becomes

$$f_{Sopt} = -\mathrm{J}_0(kr)f_P, \tag{13}$$

where $\mathrm{J}_0()$ is the zeroth order Bessel function of the first kind. The available power reduction given by equation (12) is then [20]

$$W_{T\,min}/W_{PP} = 1 - \mathrm{J}_0^2(kr_0), \tag{14}$$

which is plotted in Figure 1, and where W_{PP}, defined by equation (11), becomes $W_{PP} = \frac{1}{2}\beta_0 f_P^2$. It is important also to consider the analogous results for other kinds of source excitation.

2.2. A SET OF PRIMARY MOMENTS OR FORCES CONTROLLED, RESPECTIVELY, BY A SET OF SECONDARY FORCES OR BY A SET OF SECONDARY MOMENTS

The first case studied consists of an infinite plate excited by a set of primary moments $\mathbf{m}_P$ with a given orientation. The goal is to control the power input due to a primary moment array $\mathbf{m}_P$ by a set of secondary forces $\mathbf{f}_S$ applied on the plate. This is achieved by considering the power supplied to the plate by both the force and moment arrays. For the set of secondary forces $\mathbf{f}_S$, the power supplied to the plate can be written in the form

$$W_G = \tfrac{1}{2}\,\mathrm{Re}\,\{\mathbf{f}_S^{\mathrm{H}}\,\mathbf{v}_S\}, \tag{15}$$

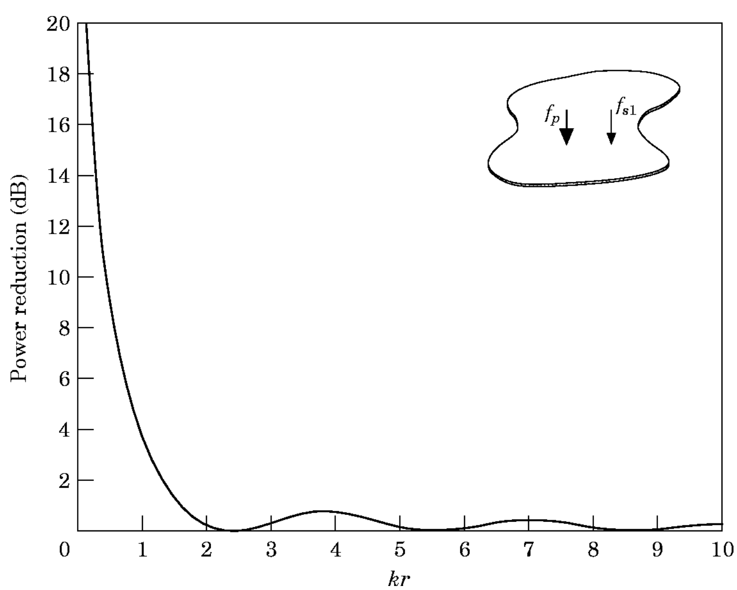

Figure 1. Power reduction against kr in the case of a primary force controlled by a secondary force by using total power minimization on an infinite plate.

where $\mathbf{v}_S$ is the linear velocity of the secondary force application points. For the set of complex moments in the vector $\mathbf{m}_P$, the power input to the plate can also be written as

$$W_M = \tfrac{1}{2}\,\mathrm{Re}\,\{\mathbf{m}_P^{\mathrm{H}}\,\mathbf{w}_P\}, \tag{16}$$

where $\mathbf{w}_P$ is the vector of complex angular velocities at the points at which the moments are applied. The total power is simply the sum of the two power contributions

$$W_T = W_F + W_M. \tag{17}$$

It has been seen in section 2.1 that the velocity $\mathbf{v}_S$ is related to the force $\mathbf{f}_S$ by the "force" mobility matrix $\mathbf{M}$. In the same manner, one can also define a "moment" mobility matrix $\mathbf{P}$ which links the set of angular velocities $\mathbf{w}_P$ and the set of moments $\mathbf{m}_P$. But one also has to consider the case of an angular velocity induced by a point force, via the matrix $\mathbf{M}'_{PS}$, and a linear velocity induced by a moment via the matrix $\mathbf{P}'_{SP}$. Hence, the vectors of linear and angular velocities due to both force and moment vectors [25] can be written as

$$\mathbf{v}_S = \mathbf{M}_{SS}\,\mathbf{f}_S + \mathbf{P}'_{SP}\,\mathbf{m}_P \qquad \text{and} \qquad \mathbf{w}_P = \mathbf{M}'_{PS}\,\mathbf{f}_S + \mathbf{P}_{PP}\,\mathbf{m}_P. \tag{18}$$

Then equation (17) becomes

$$W_T = \tfrac{1}{2}\,\mathrm{Re}\,\{\mathbf{f}_S^{\mathrm{H}}\,\mathbf{M}_{SS}\,\mathbf{f}_S + \mathbf{f}_S^{\mathrm{H}}\,\mathbf{P}'_{SP}\,\mathbf{m}_P + \mathbf{m}_P^{\mathrm{H}}\,\mathbf{M}'_{PS}\,\mathbf{f}_S + \mathbf{m}_P^{\mathrm{H}}\,\mathbf{P}_{PP}\,\mathbf{m}_P\}. \tag{19}$$

At this stage, an important question arises about the assumption of reciprocity used in the previous section. For an infinite plate, one can assume full reciprocity, i.e., between the set of forces itself ($\mathbf{M}_{SS} = \mathbf{M}_{SS}^{\mathrm{T}}$), between the set of moments itself ($\mathbf{P}_{PP} = \mathbf{P}_{PP}^{\mathrm{T}}$), and, because of the symmetry on the infinite plate, between the set of forces and the set of moments ($\mathbf{P}'_{SP} = \mathbf{P}'^{\mathrm{T}}_{PS} = \mathbf{M}'^{\mathrm{T}}_{PS}$ and $\mathbf{M}'^{\mathrm{T}}_{SP} = \mathbf{M}'^{\mathrm{T}}_{PS} = \mathbf{P}'^{\mathrm{T}}_{PS}$). On a finite plate not all of these assumptions generally hold. If, for generality, no assumption is made about reciprocity, equation (19) can be written as

$$W_T = \tfrac{1}{2}\,\{\mathbf{f}_S^{\mathrm{H}}\,\mathbf{R}_{SS}\,\mathbf{f}_S + \mathbf{m}_P^{\mathrm{H}}\,\mathbf{S}_{PP}\,\mathbf{m}_P^{\mathrm{H}} + \tfrac{1}{2}\,[(\mathbf{M}'_{PS} + \mathbf{P}'^{\mathrm{H}}_{SP})\mathbf{f}_S + \mathbf{f}_S^{\mathrm{H}}\,(\mathbf{M}'^{\mathrm{H}}_{PS} + \mathbf{P}'_{SP})\mathbf{m}_P]\}, \tag{20}$$

where $\mathbf{M}_{SS} = \mathbf{R}_{SS} + \mathrm{j}\mathbf{X}_{SS}$, $\mathbf{M}'_{PS} = \mathbf{R}'_{PS} + \mathbf{j}\mathbf{X}'_{PS}$, $\mathbf{P}'_{SP} = \mathbf{S}'_{SP} + \mathrm{j}\mathbf{Y}'_{SP}$, and $\mathbf{P}_{PP} = \mathbf{S}_{PP} + \mathrm{j}\mathbf{Y}_{PP}$. W_T is a quadratic function of $\mathbf{f}_S$ and is minimized by setting

$$\mathbf{f}_S = \mathbf{f}_{Sopt} = -\tfrac{1}{2}\,\mathbf{R}_{SS}^{-1}\,((\mathbf{M}'^{\mathrm{H}}_{PS} + \mathbf{P}'_{SP})\mathbf{m}_P. \tag{21}$$

Inserting the optimal solution given by equation (21) into equation (20) leads to the expression for the maximum power reduction:

$$\frac{W_{T\,min}}{W_{PP}} = 1 - \frac{1}{4}\,\frac{\mathbf{m}_P^{\mathrm{H}}\,(\mathbf{M}'_{PS} + \mathbf{P}'^{\mathrm{H}}_{SP})\mathbf{R}_{SS}^{-1}\,(\mathbf{M}'^{\mathrm{H}}_{PS} + \mathbf{P}'_{SP})\mathbf{m}_P}{\mathbf{m}_P^{\mathrm{H}}\,\mathbf{S}_{PP}\,\mathbf{m}_P}. \tag{22}$$

Even if full reciprocity is not assumed in equation (22), but $\mathbf{P}'_{SP} = \mathbf{M}'^{\mathrm{T}}_{PS}$ and $\mathbf{M}'_{SP} = \mathbf{P}'^{\mathrm{T}}_{PS}$, which are valid for both the infinite and the finite case, equation (21) can be simplified as

$$\mathbf{f}_S = \mathbf{f}_{Sopt} = -\mathbf{R}_{SS}^{-1}\,\mathbf{S}'_{SP}\,\mathbf{m}_P, \tag{23}$$

and the available power reduction given by equation (22) is reduced to

$$W_{T\,min}\,/W_{PP} = 1 - \mathbf{m}_P^{\mathrm{H}}\,\mathbf{S}'_{SP}\,\mathbf{R}_{SS}^{-1}\,\mathbf{R}'_{PS}\,\mathbf{m}_P\,/\mathbf{m}_P^{\mathrm{H}}\,\mathbf{S}_{PP}\,\mathbf{m}_P. \tag{24}$$

Because of the symmetry of the equations, equation (24) also represents the reduction in the total power input if a set of primary forces $\mathbf{f}_P$ were optimally controlled with a set of secondary moments with a given orientation on an infinite plate.

If one considers a single primary force on an infinite plate $\mathbf{f}_P = f_P$ controlled by a single secondary moment a distance r away and oriented at an angle θ_S to the radial direction

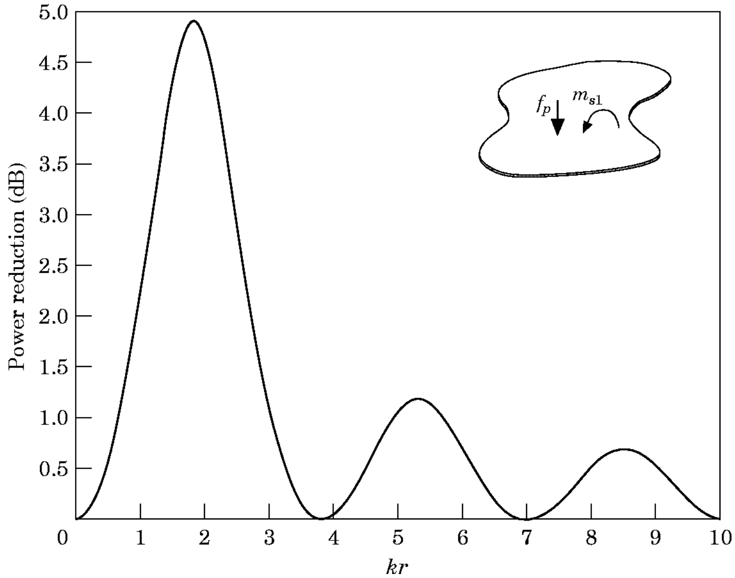

Figure 2. Power reduction against kr in the case of a primary force controlled by a secondary moment by using total power minimization on an infinite plate.

$\mathbf{m}_s = m_s$ and, as is shown in Appendix A takes $\mathbf{R}_{PP} = \beta_0$, and $\mathbf{S}_{SS} = \gamma_0 = (k^2/2)\beta_0$ and $\mathbf{S}'_{SP} = \mathbf{R}'_{PS} = k\beta_0\,\mathbf{J}_1\,(kr)\cos\theta_s$, the maximum power reduction is found to be given by

$$W_{T\,min}\,/W_{PP} = 1 - 2\mathrm{J}_1^2\,(kr)\cos^2\theta_S, \tag{25}$$

where $W_{PP} = \beta_0 f_P^2$ is the power output of the primary moment without any control and J_1 () is the first order Bessel function of the first kind.

The attenuation in the total power transmitted to the plate, given by equation (25), is plotted as a function of (kr) in Figure 2 for $\theta_S = 0$. It is clear that a maximum of about 5 dB reduction can be achieved when controlling a force with a secondary moment on an infinite plate. This is not a large reduction compared with the control of a point force by another force, for which greater than 60 dB reduction can, in principle, be achieved. No reduction is obtained when the force and moment are collocated, because the secondary moment cannot then affect the linear velocity at the position of the primary source.

It should be noted that when a set of primary forces is controlled by a set of secondary moments the control performance is also dependent on the orientation of the control moments. Thus, in general, better control performance can be achieved when a new cost function which calculates both the optimal strength (amplitude and phase) and the optimal orientation of the control moments is minimized. For the case plotted in Figure 2, it was clear that the optimal alignment of the secondary moment was in the direction of the segment jointing the points at which the primary force and control moment are applied. In fact, the vibrations generated by the control moment have a dipole-line spatial distribution, which has a maximum amplitude along the segment which aligns the moment with the primary force.

Most control actuators that produce a moment do so with a fixed orientation. By using two collocated moment actuators oriented orthogonal to each other, it would be possible to generate a moment with an arbitrary orientation. This can be achieved by driving the two actuators in phase and by adjusting their amplitudes such that the resulting moment has the required orientation and amplitude. Therefore, the three control parameters for this type of actuator are the amplitudes of each moment source and the phase, which is equal for both sources.

2.3. A SET OF PRIMARY MOMENTS CONTROLLED BY A SET OF SECONDARY MOMENTS

The process of calculating the power input in this case is exactly the same as in Section 2.1. Considered here is an infinite plate excited by a set of primary moments $\mathbf{m}_P$ with a given orientation controlled by a set of secondary moments $\mathbf{m}_S$ the orientation of which is also given. Upon assuming reciprocity and replacing $\mathbf{f}_S$ by $\mathbf{m}_S$, $\mathbf{f}_P$ by $\mathbf{m}_P$ and $\mathbf{R}$ by $\mathbf{S}$, the expression for the total power output of both arrays of moments becomes

$$W_T = \tfrac{1}{2}\,[\mathbf{m}_P^{\mathrm{H}}\,\mathbf{S}_{PP}\,\mathbf{m}_P + \mathbf{m}_S^{\mathrm{H}}\,\mathbf{S}_{SP}\,\mathbf{m}_P + \mathbf{m}_P^{\mathrm{H}}\,\mathbf{S}_{PS}\,\mathbf{m}_S + \mathbf{m}_S^{\mathrm{H}}\,\mathbf{S}_{SS}\,\mathbf{m}_S], \tag{26}$$

where $\mathbf{S}$ is the real part of the angular mobility matrix (which is analogous to $\mathbf{R}$, the real part of the linear mobility matrix). If equation (26) is minimized with respect to the real and imaginary parts of the elements of $\mathbf{m}_S$, the optimal set of secondary moments is found to be

$$\mathbf{m}_S = \mathbf{m}_{Sopt} = -\mathbf{S}_{SS}^{-1}\,\mathbf{S}_{SP}\,\mathbf{m}_P. \tag{27}$$

The ratio of the total power output with and without control, which is also the attenuation in the total power transmitted to the plate when the optimal set of secondary moments is applied on the plate, is thus given by

$$W_{T\,min}/W_{PP} = 1 - \mathbf{m}_P^{\mathrm{H}}\,\mathbf{S}_{PS}\,\mathbf{S}_{SS}^{-1}\,\mathbf{S}_{SP}\,\mathbf{m}_P/\mathbf{m}_P^{\mathrm{H}}\,\mathbf{S}_{PP}\,\mathbf{m}_P. \tag{28}$$

In the case of a single primary moment controlled by a single moment aligned in the source direction on an infinite plate $\mathbf{m}_S = m_S$, $\mathbf{m}_P = m_P$, and for an infinite plate it is shown in Appendix A that $\mathbf{S}_{SS} = \mathbf{S}_{PP} = \gamma_0$ and $\mathbf{S}_{SP} = \mathbf{S}_{PS} = 2\gamma_0\,\{\mathrm{J}_0(kr) - (1/kr)\mathrm{J}_1(kr)]$ (see equation A16) so that equation (27) becomes

$$m_{Sopt}/m_p = -2[\mathrm{J}_0(kr) - (1/kr)\mathrm{J}_1(kr)] \tag{29}$$

and the power reduction given by equation (28) is thus

$$W_{T\,min}/W_{PP} = 1 - 4[\mathrm{J}_0(kr) - (1/kr)\mathrm{J}_1(kr)]^2 \tag{30}$$

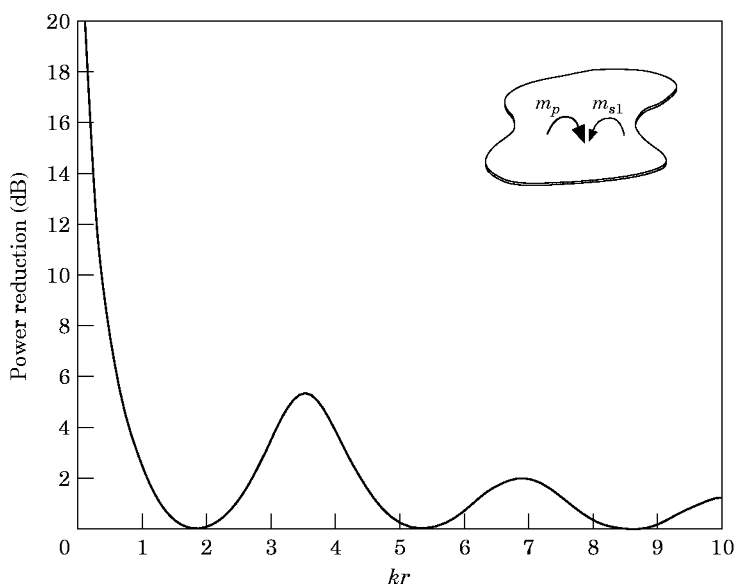

Figure 3. Power attenuation against kr in the case of a primary moment controlled by a secondary moment by using total power minimization on an infinite plate.

which is plotted in Figure 3. Note that reduction is again greatest for small separations, as in Figure 1, but that somewhat larger attenuations than those shown in Figure 1 can be achieved when $kr \approx 2{\cdot}4$, $5{\cdot}5$, $8{\cdot}6, \ldots,$ etc.

2.4. A PRIMARY FORCE CONTROLLED BY A COLLOCATED SECONDARY FORCE AND A SECONDARY MOMENT

It may be possible to design a secondary actuator which is able to produce independently a control force f_S and a control moment m_S with a given orientation.

If these two collocated secondary sources f_S and m_S excite an infinite plate, then the response of the plate at the point at which f_S and m_S are acting is characterized by both linear velocity v_S and angular velocity w_S. The power input by the two sources is then given by $W_{f_S m_S} = (1/2)\,\mathrm{Re}\,(f_s^*\, v_s + m_s^*\, w_s)$ and, by considering equation (18), it can be written as $W_{f_S m_S} = (1/2)\,\mathrm{Re}\,(f_S^*\, M_{SS} f_S + f_S^*\, P'_{SS}\, m_S + m_S^*\, M'_{SS} f_S + m_S^*\, P_{SS}\, m_S)$. However, for the infinite plate there are no coupling terms in the point-mobility which link either the linear velocity to a moment excitation ($v_S/m_S = P'_{SS} = 0$) or the angular velocity to a force excitation ($w_S/f_S = M'_{SS} = 0$) [25]. The total power supplied by the two collocated sources is thus given by the sum of the power supplied by the force and the moment when acting alone so that $W_{f_S m_S} = 1/2(f_S^*\, R_{SS} f_S + m_S^*\, S_{SS}\, m_S)$. As a consequence of this it has been found that the optimal control force and moment which simultaneously minimize the total power input when acting together are the same as those calculated in section 2.1 for a secondary force f_S and in section 2.2 for a secondary moment m_S, when acting alone. In other words, the collocated control force and moment can be driven independently in order to achieve the best control. The power reduction in total power supplied to the plate when the secondary force and moment are acting together is given by $W_{T\,min}/W_{PP} = 1 - \Delta_{f_P f_S} - \Delta_{f_P m_S}$, where $\Delta_{f_P f_S} = f_P^*\, R_{PS}\, R_{SS}^{-1}\, R_{SP} f_P / f_P^*\, R_{PP} f_P$ is the fractional reduction if the control force f_S is acting alone (equation (12)) and $\Delta_{f_P m_S} = f_P^*\, R'_{PS}\, R_{SS}^{-1}\, S'_{SP} f_P / f_P^*\, R_{PP} f_P$ is the fractional reduction if the control moment m_S is acting alone (equation (24)). Because both fractional reductions are positive for any value of kr, the power reduction of the collocated control force and moment is always greater than or equal to that obtained for a single control force or for a single control moment. Moreover, $W_{T\,min}/W_{PP}$ must always be positive, even when the total power input is minimized. This means that the action of the two collocated control sources is inevitably matched, in such a way that when the control force is very effective the control moment performs poorly and *vice versa*. In Figure 4 is shown the power reduction when a collocated secondary force and moment, positioned in a vertical plane oriented in the radial direction, are used.

2.5. A PRIMARY FORCE CONTROLLED BY A SET OF SECONDARY SOURCES DISTRIBUTED IN A RING AROUND THE PRIMARY SOURCE

The numerical results presented in previous sections were only for the particular cases of controlling one primary source by using a single secondary source that could be either a force, a moment or a collocated force and moment. In this section the control of one primary force by using several secondary sources arranged in a ring around the primary source is presented. A uniform distribution around the ring is assumed and two cases are considered: the first is characterized by a set of control forces (Figures 5(a–c)), while the second involves a set of collocated control forces and control moments (Figures 5(d–f)). The control moments are positioned in vertical planes oriented in the radial direction. In particular the control effectiveness of a single control source (Figures 5(a) and 5(d)) is compared with the control efficacy of two and four control sources with a ring distribution as shown in Figures 5(b), 5(c) and 5(e), 5(f).

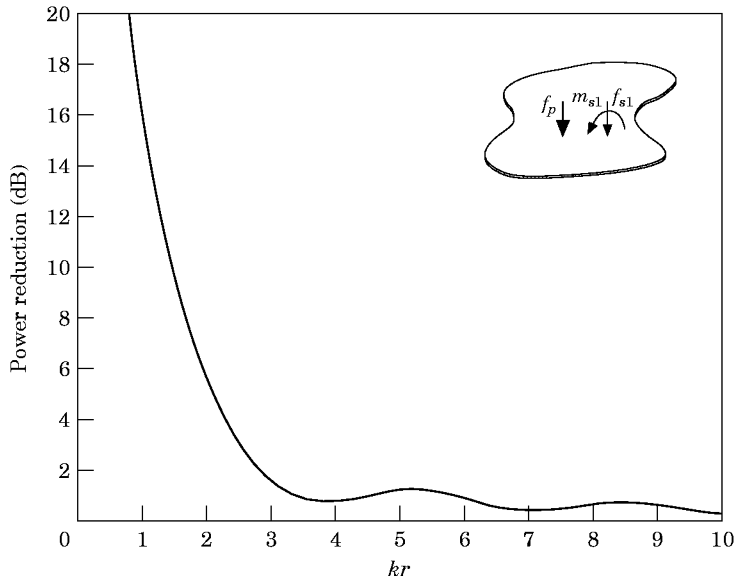

Figure 4. Power attenuation against kr in the case of a primary force controlled by a collocated secondary force and a secondary moment by using total power minimization on an infinite plate.

The matrix equations of section 2.1 were used to study the control with a ring of forces, while the analysis of the control with a ring of collocated forces and moments required an appropriate matrix formulation. In this case the power input by the secondary forces and moments cannot be calculated by considering each source acting independently, as has been assumed in section 2.4 for a single collocated secondary force and moment. However, no analytic expression for the power reduction is given here because of the complexity of the solution.

In Figure 6 are shown the power reductions, respectively, when one, two, four or eight secondary forces are used. This graph shows clearly that as the number of control sources increases the control become more effective. With one control source it is possible to

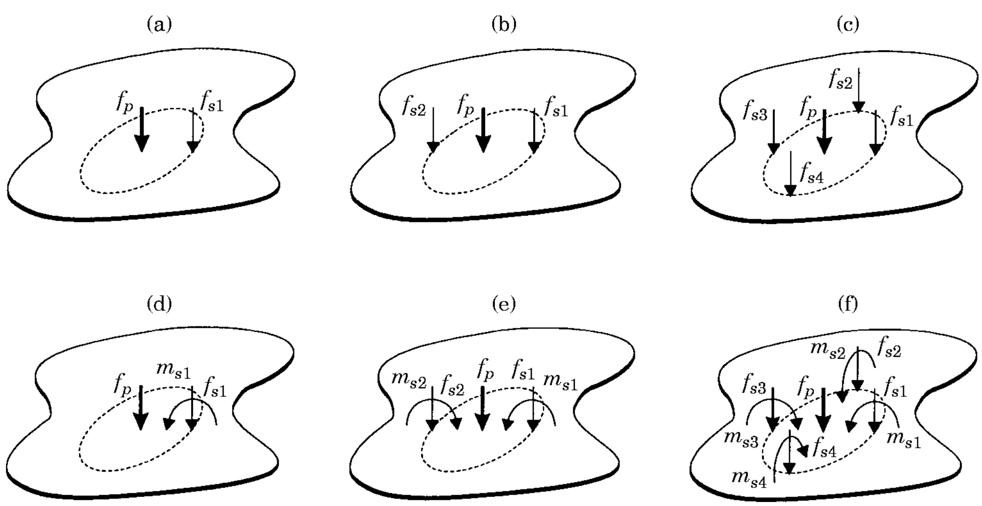

Figure 5. Ring of secondary force (a–c) or collocated force and moment (d–f) control sources acting on an infinite plate excited by a primary force.

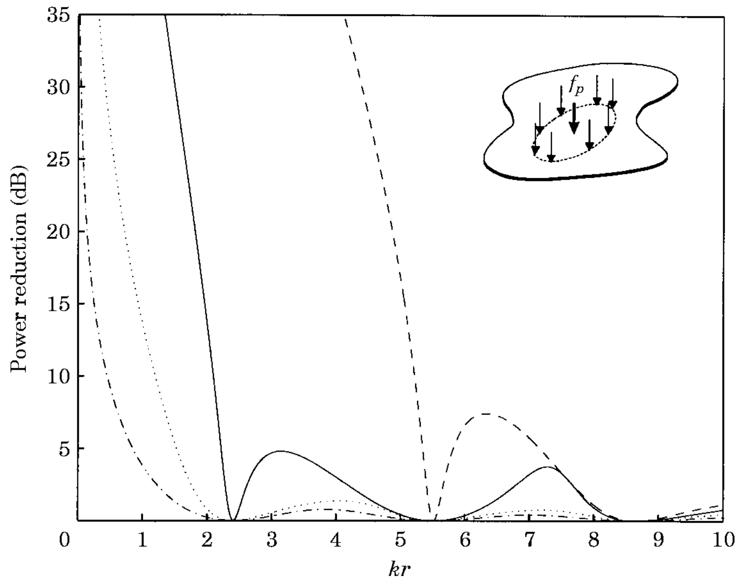

Figure 6. Power reduction against kr in the case of a primary force controlled by one (– · – · –), two (.....), four (——) and eight (– – –) secondary forces by using total power minimization on an infinite plate.

achieve a power reduction higher than 20 dB for $kr \leqslant 0{\cdot}15$, while when considering two or four control sources such a limit is achieved, respectively, for $kr \leqslant 0{\cdot}75$ and $kr \leqslant 1{\cdot}81$. Nevertheless, it is still impossible to achieve any control when $kr \approx 2{\cdot}4$, $5{\cdot}5$, $8{\cdot}6, \ldots$. This is because of the symmetry of the control sources. If the control forces were slightly moved from the perfectly symmetric configuration some control can also be achieved for these values of kr. The case of eight control forces has also been considered and the simulations have shown that very large power reductions, of at least 20 dB, can be achieved at almost all frequencies for which $kr \leqslant 5$. In fact, the dashed line for this case in Figure 6 has a dip in a very narrow band around $kr \approx 2{\cdot}4$, which is not shown in the plot because for reductions of below 35 dB this dip is narrower than the kr sampling used in the simulations. This dip can again be removed by randomly altering the positions of the secondary sources by a small amount to break the symmetry.

The results obtained for the structural case described above were compared with those obtained for a similar acoustic system composed of a primary monopole excitation and one, two, four or eight control monopoles placed at a fixed distance from the source (r) and equally distant from each other [23]. Power reductions in the structural and acoustic cases showed similar trends when one, two or four control sources were used, while the simulation with eight control sources has shown that the acoustical system produces power reductions of greater than 20 dB only up to $kr \approx 2{\cdot}3$, where k is now the acoustic wavenumber, which is much lower than the limit ($kr \leqslant 5$) found for the structural case.

In Figure 7 are shown the power reductions, respectively, when one, two or four secondary collocated forces and moments are acting. A power reduction greater than 20 dB is achieved for $kr \leqslant 0{\cdot}7$ when one force and moment are acting or for $kr \leqslant 2$ when two pairs of force and moment are acting and finally for $kr \leqslant 4{\cdot}8$ when four pairs of force and moment are used. The simultaneous action of the force and the moment produce the benefit shown in section 2.4 since, as shown in Figure 6, the power reduction changes smoothly apart from the singularity at $kr \approx 3{\cdot}3$ which can again be avoided by perturbing the positions of the secondary sources.

The results shown in Figures 6 and 7 show that the use of collocated forces and moments performs better than forces alone for the same number of independent secondary sources (for example, four secondary forces and two pairs of collocated forces and moments). Both approaches have a power reduction greater than 20 dB for $kr \leqslant 1{\cdot}81$, but, for $kr > 1{\cdot}81$, the power reduction curve decreases smoothly for the collocated force and moment control and produces a power reduction higher than 5 dB up to $kr = 3$, while the power reduction of the other control approach decreases drastically and reaches zero for $kr \approx 2{\cdot}4$.

Several simulations were run in which different types of primary excitation were considered. In particular, the case of a primary force applied some distance from the center of the ring of control forces but still within it was considered. The simulations showed that both the ring of forces or the ring of collocated forces and moments cannot perform as well as in the symmetric case. Moreover, the control sources assume different amplitudes and phases, while in the symmetric case they were all driven with the same strength. A second interesting case was considered in which the primary source was a collocated point force and moment oriented at 30° with respect to one of the axes of the four control sources. Also in this case a lower active control performance, in comparison to the case in which only a primary force excited the system, was achieved. Because the primary moment was not aligned with any of the secondary moments each of the control sources had a different amplitude and phase even with the primary source placed at the centre of the ring of control sources.

3. MAXIMIZING POWER ABSORPTION

3.1. A SET OF PRIMARY FORCES CONTROLLED BY A SET OF SECONDARY FORCES

An alternative strategy for active control is the maximization of power absorption by the set of secondary forces. This is equivalent to minimizing the power output of the set of secondary forces $\mathbf{f}_S$ alone. The power output of the set of secondary forces can be written as

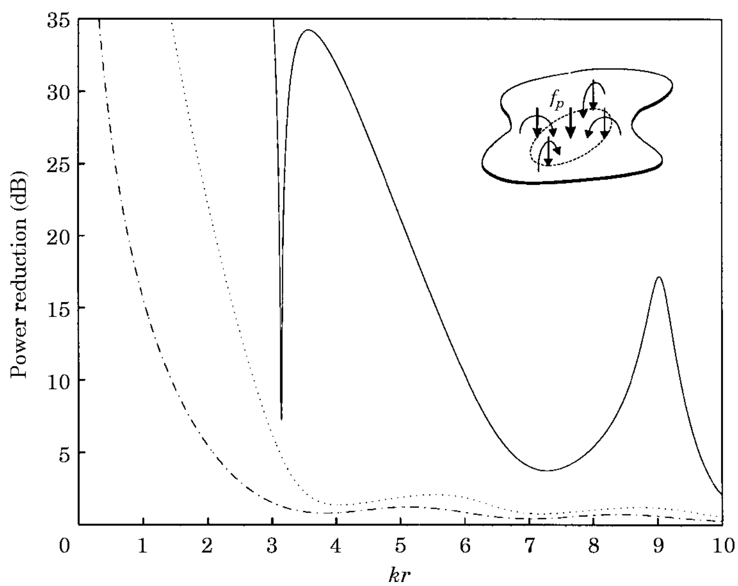

Figure 7. Power reduction against kr in the case of a primary force controlled by one (– · – · –), two (.....) and four (——) secondary forces and moments by using total power minimization on an infinite plate.

$$W_S = \tfrac{1}{2}\,\mathrm{Re}\,\{\mathbf{f}_S^{\mathrm{H}}\,\mathbf{v}_S\} = \tfrac{1}{2}\,\mathrm{Re}\,\{\mathbf{f}_S^{\mathrm{H}}\,\mathbf{M}_{SS}\,\mathbf{f}_S + \mathbf{f}_S^{\mathrm{H}}\,\mathbf{M}_{SP}\,\mathbf{f}_P\}, \tag{31}$$

which in turn can be written, after some manipulation, as

$$W_S = \tfrac{1}{2}\,\mathbf{f}_S^{\mathrm{H}}\,\mathbf{R}_{SS}\,\mathbf{f}_S + \tfrac{1}{4}\,\mathbf{f}_S^{\mathrm{H}}\,\mathbf{M}_{SP}\,\mathbf{f}_P + \tfrac{1}{4}\,\mathbf{f}_P^{\mathrm{H}}\,\mathbf{M}_{SP}^{\mathrm{H}}\,\mathbf{f}_S. \tag{32}$$

Equation (32) is minimized by

$$\mathbf{f}_S = \mathbf{f}_{Sa} = -\tfrac{1}{2}\,\mathbf{R}_{SS}^{-1}\,\mathbf{M}_{SP}\,\mathbf{f}_P. \tag{33}$$

Inserting this optimally absorbing solution into equation (32) leads to the minimum power output of the set of secondary forces:

$$W_{Sa} = -\tfrac{1}{8}\,\mathbf{f}_P^{\mathrm{H}}\,\mathbf{M}_{SP}^{\mathrm{H}}\,\mathbf{R}_{SS}^{-1}\,\mathbf{M}_{SP}\,\mathbf{f}_P. \tag{34}$$

One knows that the total power output is given by

$$W_T = \tfrac{1}{2}\,[\mathbf{f}_P^{\mathrm{H}}\,\mathbf{R}_{PP}\,\mathbf{f}_P + \mathbf{f}_P^{\mathrm{H}}\,\mathbf{R}_{PS}\,\mathbf{f}_S + \mathbf{f}_S^{\mathrm{H}}\,\mathbf{R}_{SP}\,\mathbf{f}_P + \mathbf{f}_S^{\mathrm{H}}\,\mathbf{R}_{SS}\,\mathbf{f}_S]. \tag{35}$$

By using $W_{PP} = \frac{1}{2}\,\mathbf{f}_P^{\mathrm{H}}\,\mathbf{R}_{PP}\,\mathbf{f}_P$ as the power input to the plate before control the power reduction due to the optimally absorbing solution can be calculated by substituting equation (33) into equation (35) which leads to the following expression:

$$\frac{W_{Ta}}{W_{PP}} = 1 - \frac{1}{2}\,\frac{[\mathbf{f}_P^{\mathrm{H}}\,\mathbf{R}_{SP}^{\mathrm{T}}\,\mathbf{R}_{SS}^{-1}\,\mathbf{M}_{SP}\,\mathbf{f}_P + \mathbf{f}_P^{\mathrm{H}}\,\mathbf{M}_{SP}^{\mathrm{H}}\,\mathbf{R}_{SS}^{-1}\,\mathbf{R}_{SP}\,\mathbf{f}_P - \frac{1}{2}\,\mathbf{f}_P^{\mathrm{H}}\,\mathbf{M}_{SP}^{\mathrm{H}}\,\mathbf{R}_{SS}^{-1}\,\mathbf{M}_{SP}\,\mathbf{f}_P]}{\mathbf{f}_P^{\mathrm{H}}\,\mathbf{R}_{PP}\,\mathbf{f}_P}. \tag{36}$$

Consider first the simple case of a single primary force acting on an infinite plate and controlled by a single secondary force by using the maximization of its power absorption. One has $\mathbf{f}_P = f_P$, $\mathbf{f}_S = f_S$, and, from Appendix A, $\mathbf{R}_{PP} = \mathbf{R}_{SS} = \beta_0$ and $\mathbf{M}_{SP} = \beta_0\,[\mathrm{J}_0\,(kr) - \mathrm{j}(\mathrm{Y}_0\,(kr) + (2/\pi)\mathrm{K}_0\,(kr))]$. From equation (36), with $W_{PP} = \frac{1}{2}\,\beta_0 f_P^2$, which is the total power without any control, the available power reduction can be written in the form

$$W_{Ta}\,/W_{PP} = 1 - \tfrac{3}{4}\,\mathrm{J}_0^2\,(kr) + \tfrac{1}{4}\,(\mathrm{Y}_0\,(kr) + (2/\pi)\mathrm{K}_0\,(kr))^2. \tag{37}$$

The reduction in total power supplied to the plate when this strategy is implemented is plotted in Figure 8.

It is very interesting to note that, even in the limit $kr \to 0$, the power supplied to the receiving structure remains finite in this structural example and is even reduced by 6 dB. In a three-dimensional free field acoustic example, the power output of a primary and secondary monopole source adjusted to maximize the absorption of the secondary source becomes infinite, as $kr \to 0$ [21]. This is due to the large reactive near field of acoustic monopoles in free space. The point force acting on a plate does not have a reactive near field (the mechanical point input impedance is entirely real) and so this singularity does not occur.

3.2. A SET OF PRIMARY MOMENTS OR FORCES CONTROLLED, RESPECTIVELY, BY A SET OF SECONDARY FORCES OR BY A SET OF SECONDARY MOMENTS

In this case the plate is originally excited by a set of primary moments and one seeks to maximize the power absorption of the set of secondary forces, which is equivalent to minimizing the power output of the set of secondary forces $\mathbf{f}_S$ alone. Using the formulation of section 2.2 and considering the left side of equation (18) one has the following formulation for the power output of $\mathbf{f}_S$,

$$W_S = \tfrac{1}{2}\,\mathrm{Re}\,\{\mathbf{f}_S^{\mathrm{H}}\,\mathbf{M}_{SS}\,\mathbf{f}_S + \mathbf{f}_S^{\mathrm{H}}\,\mathbf{P}'_{SP}\,\mathbf{m}_P\}, \tag{38}$$

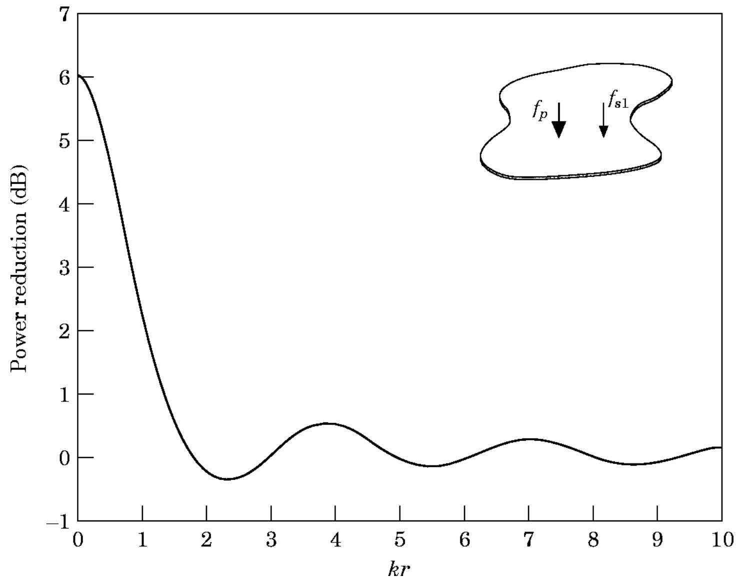

Figure 8. Power reduction against kr in the case of a primary force controlled by a secondary force by using maximization of the secondary source power absorption on an infinite plate.

which can be written as in section 3.1 in the form

$$W_S = \tfrac{1}{2}\,\mathbf{f}_S^{\mathrm{H}}\,\mathbf{R}_{SS}\,\mathbf{f}_S + \tfrac{1}{4}\,\mathbf{f}_S^{\mathrm{H}}\,\mathbf{P}'_{SP}\,\mathbf{m}_P + \tfrac{1}{4}\,\mathbf{m}_P^{\mathrm{H}}\,\mathbf{P}'^{\mathrm{H}}_{SP}\,\mathbf{f}_S. \tag{39}$$

Equation (39) is minimized by setting

$$\mathbf{f}_S = \mathbf{f}_{Sa} = -\tfrac{1}{2}\,\mathbf{R}_{SS}^{-1}\,\mathbf{P}'_{SP}\,\mathbf{m}_P. \tag{40}$$

Inserting the solution given by equation (40) into equation (39) gives the minimum of the power output of the set of secondary forces:

$$W_{Sa} = -\tfrac{1}{8}\,\mathbf{m}_P^{\mathrm{H}}\,\mathbf{P}'^{\mathrm{H}}_{SP}\,\mathbf{R}_{SS}^{-1}\,\mathbf{P}'\mathbf{m}_P. \tag{41}$$

The total power output is given by

$$W_T = \tfrac{1}{2}\,[\mathbf{m}_P^{\mathrm{H}}\,\mathbf{S}_{PP}\,\mathbf{m}_P + \mathbf{m}_P^{\mathrm{H}}\,\mathbf{S}'^{\mathrm{T}}_{SP}\,\mathbf{f}_S + \mathbf{f}_S^{\mathrm{H}}\,\mathbf{S}'_{SP}\,\mathbf{m}_P + \mathbf{f}_S^{\mathrm{H}}\,\mathbf{R}_{SS}\,\mathbf{f}_S]. \tag{42}$$

The available power reduction is calculated by inserting the solution of equation (40) into equation (42) and, since the power input before control is $W_{PP} = \tfrac{1}{2}\,\mathbf{m}_P^{\mathrm{H}}\,\mathbf{S}_{PP}\,\mathbf{m}_P$, one finds

$$\frac{W_{Ta}}{W_{PP}} = 1 - \frac{1}{2}\,\frac{[\mathbf{m}_P^{\mathrm{H}}\,\mathbf{S}'^{\mathrm{T}}_{SP}\,\mathbf{R}_{SS}^{-1}\,\mathbf{P}'_{SP}\,\mathbf{m}_P + \mathbf{m}_P^{\mathrm{H}}\,\mathbf{P}'^{H}_{SP}\,\mathbf{R}_{SS}^{-1}\,\mathbf{S}'_{SP}\,\mathbf{m}_P - \tfrac{1}{2}\,\mathbf{m}_P^{\mathrm{H}}\,\mathbf{P}'^{\mathrm{H}}_{SP}\,\mathbf{R}_{SS}^{-1}\,\mathbf{P}'_{SP}\,\mathbf{m}_P]}{\mathbf{m}_P^{\mathrm{H}}\,\mathbf{S}_{PP}\,\mathbf{m}_P}. \tag{43}$$

The case of one primary force controlled by one secondary moment is not treated because of the exact similarity with the present case. All of the results are the same except that one exchanges $\mathbf{m}_P$ with $\mathbf{f}_P$ and $\mathbf{f}_S$ with $\mathbf{m}_S$.

In the case of a single primary force controlled by a single secondary moment a distance r away and positioned in a vertical plane oriented in the radial direction, one has $\mathbf{m}_s = m_s$, $\mathbf{f}_p = f_p$, and, from Appendix A, $\mathbf{R}_{PP} = \beta_0$, $\mathbf{S}_{SS} = \gamma_0 = (k^2/2)\beta_0$ and $\mathbf{P}'_{PS} = k\beta_0\,[\mathrm{J}_1(kr) - \mathrm{j}(\mathrm{Y}_1(kr) + (2/\pi)\mathrm{K}_1(kr))]$. Hence, with $W_P = \beta_0 f_P^2$, equation (43) becomes

$$W_{Ta}/W_{PP} = 1 - \tfrac{3}{2}\,\mathrm{J}_1^2(kr) + \tfrac{1}{2}\,(\mathrm{Y}_1(kr) + (2/\pi)\mathrm{K}_1(kr))^2. \tag{44}$$

The reduction in total power supplied to the plate when this strategy is adopted is plotted in Figure 9. Although small increases in total power are observed for certain separations, the effect of this control strategy on an infinite plate is generally to reduce the level of plate excitation, by about half the amount which can be achieved by minimizing the power in this case (see Figure 2).

3.3. A SET OF PRIMARY MOMENTS CONTROLLED BY A SET OF SECONDARY MOMENTS

The process is directly analogous to that in section 3.1. The control strategy here is to maximize the power absorption of a set of secondary moments, which is equivalent to minimizing the power output of the set of secondary moments. Consider first the case of a set of moments $\mathbf{m}_P$ for the primary excitation. The power output of the set of secondary moments $\mathbf{m}_S$ can be written as

$$W_S = \tfrac{1}{2}\,\mathrm{Re}\,\{\mathbf{m}_S^{\mathrm{H}}\,\mathbf{P}_{SS}\,\mathbf{m}_S + \mathbf{m}_S^{\mathrm{H}}\,\mathbf{P}_{SP}\,\mathbf{m}_P\}, \qquad (45)$$

which can also be written, after some manipulations, as

$$W_S = \tfrac{1}{2}\,\mathbf{m}_S^{\mathrm{H}}\,\mathbf{S}_{SS}\,\mathbf{m}_S + \tfrac{1}{4}\,\mathbf{m}_S^{\mathrm{H}}\,\mathbf{P}_{SP}\,\mathbf{m}_P + \tfrac{1}{4}\,\mathbf{m}_P^{\mathrm{H}}\,\mathbf{P}_{SP}^{\mathrm{H}}\,\mathbf{m}_S, \qquad (46)$$

where $\mathbf{S}$ is defined in section 2.2. Equation (46) is minimized by setting

$$\mathbf{m}_S = \mathbf{m}_{Sa} = -\tfrac{1}{2}\,\mathbf{S}_{SS}^{-1}\,\mathbf{P}_{SP}\,\mathbf{m}_P, \qquad (47)$$

which gives the minimum of equation (46) as

$$W_{Sa} = -\tfrac{1}{8}\,\mathbf{m}_P^{\mathrm{H}}\,\mathbf{P}_{SP}^{\mathrm{H}}\,\mathbf{S}_{SS}^{-1}\,\mathbf{P}_{SP}\,\mathbf{m}_P. \qquad (48)$$

One can then calculate the total power output:

$$W_T = \tfrac{1}{2}\,[\mathbf{m}_P^{\mathrm{H}}\,\mathbf{S}_{PP}\,\mathbf{m}_P + \mathbf{m}_P^{\mathrm{H}}\,\mathbf{S}_{SP}^{\mathrm{T}}\,\mathbf{m}_S + \mathbf{m}_S^{\mathrm{H}}\,\mathbf{S}_{SP}\,\mathbf{m}_P + \mathbf{m}_S^{\mathrm{H}}\,\mathbf{S}_{SS}\,\mathbf{m}_S]. \qquad (49)$$

The available power reduction is calculated by inserting the optimal solution given by equation (47) into equation (49). With $W_{PP} = \tfrac{1}{2}\,\mathbf{m}_P^{\mathrm{H}}\,\mathbf{S}_{PP}\,\mathbf{m}_P$, it is thus expressed by

$$\frac{W_{Ta}}{W_{PP}} = 1 - \frac{1}{2}\,\frac{[\mathbf{m}_P^{\mathrm{H}}\,\mathbf{S}_{SP}^{\mathrm{T}}\,\mathbf{S}_{SS}^{-1}\,\mathbf{P}_{SP}\,\mathbf{m}_P + \mathbf{m}_P^{\mathrm{H}}\,\mathbf{P}_{SP}^{\mathrm{H}}\,\mathbf{S}_{SS}^{-1}\,\mathbf{S}_{SP}\,\mathbf{m}_P - \tfrac{1}{2}\,\mathbf{m}_P^{\mathrm{H}}\,\mathbf{P}_{SP}^{\mathrm{H}}\,\mathbf{S}_{SS}^{-1}\,\mathbf{P}_{SP}\,\mathbf{m}_P]}{\mathbf{m}_P^{\mathrm{H}}\,\mathbf{S}_{PP}\,\mathbf{m}_P}. \qquad (50)$$

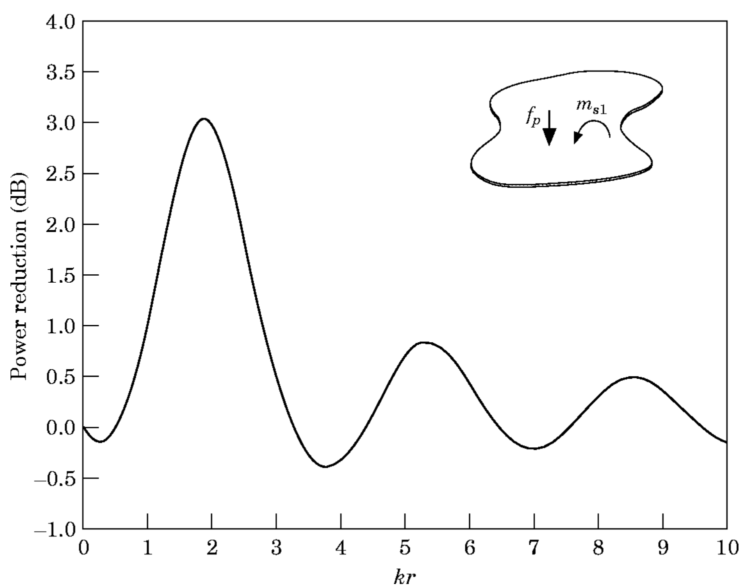

Figure 9. Power reduction against *kr* in the case of a primary force controlled by a secondary force by using maximization of the secondary source power absorption on an infinite plate.

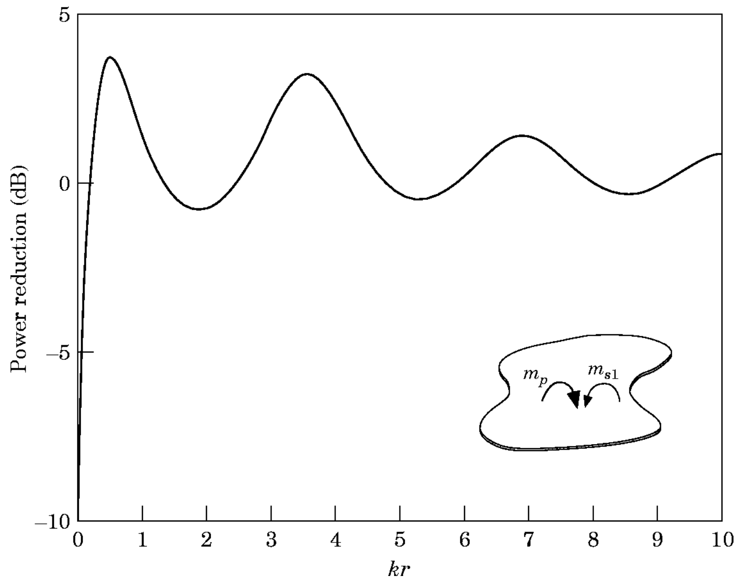

Figure 10. Power reduction against kr in the case of a primary force controlled by a secondary force by using maximization of the secondary source power absorption on an infinite plate.

In the case of a single primary moment controlled by a single aligned moment on an infinite plate, one has, $\mathbf{m}_P = \mathbf{m}_P$, $\mathbf{m}_S = m_S$, $\mathbf{S}_{SS} = \mathbf{S}_{PP} = \gamma_0$ and P_{SP} given by equation (A16). Hence equation (50) becomes

$$\frac{W_{Ta}}{W_{PP}} = 1 - 3\left(\mathrm{J}_0(kr) - \frac{1}{kr}\mathrm{J}_1(kr)\right)^2 + \left(\mathrm{Y}_0(kr) - \frac{1}{kr}\mathrm{Y}_1(kr) - \frac{2}{\pi}\left(\mathrm{K}_0(kr) + \frac{1}{kr}K_1(kr)\right)\right)^2. \tag{51}$$

Relation (51) is plotted on Figure 10. In this case, there is a singularity as $kr \to 0$ in which case the total power supplied to the plate is enormously amplified. This is due to the large reactive near field of the moment. The power supplied to the plate is only increased if kr is less than about 0·2, however, which is only a very local effect, and generally this control strategy appears beneficial even for two moments on an infinite plate, provided that their separation is greater than about $0{\cdot}03\lambda$.

3.4. A PRIMARY FORCE CONTROLLED BY A SECONDARY FORCE AND A SECONDARY MOMENT COLLOCATED

As in section 2.4, the power reduction has also been calculated when using a secondary collocated force and moment which is oriented in the radial direction, in this case acting to maximize the power absorption of both the secondary force and moment. As seen in section 2.4, the power inputs due to these two secondary actuators are independent of each other and, as a consequence of this, the optimal control force and moment are the same as those calculated in section 3.1 for a primary force and secondary force and in section 3.2 for a primary force and a secondary moment. The collocated control force and moment can thus again be driven independently in order to achieve the best control. As a consequence of this the total power absorbed by the two control sources is equal to the sum of the power absorbed by the control force and the control moment when both are acting individually. The power reduction when a collocated secondary force and moment,

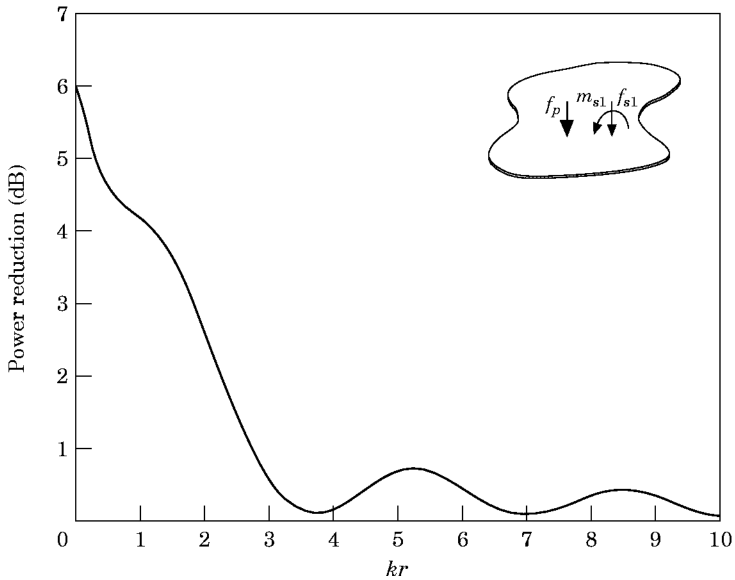

Figure 11. Power attenuation against kr in the case of a primary force controlled by a collocated secondary force and a secondary moment by using maximization of the secondary source power absorption on an infinite plate.

positioned in a vertical plane oriented in the radial direction, are used, is shown in Figure 11.

4. ACTIVE CONTROL ON A FINITE PLATE

In this section the effect is considered of the strategies for active control analyzed in sections 2 and 3 when the forces and moments act on a simply supported finite plate. In Figure 12 is shown the geometry of the finite plate: the position of the primary and secondary sources when the control action performs at a single point is shown in Figure 12(a), while the ring distribution used to study the control action by using several secondary sources is shown in Figure 12(b).

4.1 TOTAL POWER MINIMIZATION

The general solutions found in section 2 for the optimal set of secondary forces or moments and power reduction are still valid for a plate of finite size, but the expressions for the input and transfer mobilities are more complicated in this case. In order to derive

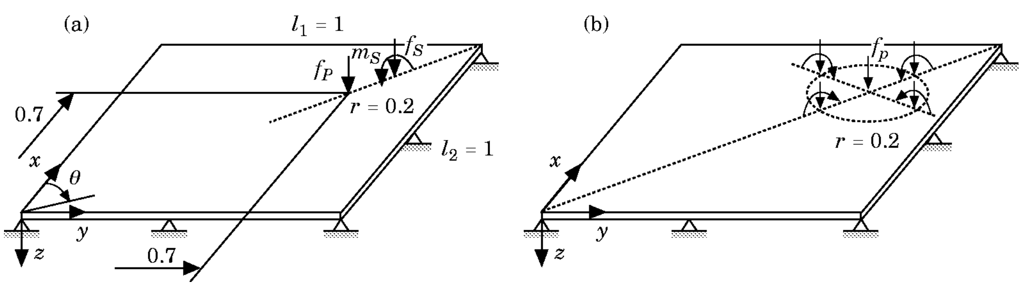

Figure 12. Geometry of the finite plate and primary and secondary sources positions: one pair of collocated force and moment control sources; (b) ring of collocated force and moment control sources.

the mobility expressions, one considers the case of a finite plate of given thickness h excited by a single primary force f_P located at (x_P, y_P). One wants to calculate the velocity induced by this excitation at any location (x, y) on the plate. The formulation for the velocity given in reference by [25] can be used and, for a simply supported plate with a uniform mass distribution m'', leads to

$$v(x_S, y_S) = \mathrm{j}\omega f_P \sum_{n=1}^{\infty} \frac{\varphi_n(x_s, y_s)\varphi_n(x_p, y_p)}{\Lambda_n[\omega_n^2(1+\mathrm{j}\eta) - \omega^2]}, \tag{52}$$

where ω is the excitation frequency, ω_n is the nth natural frequency given by $\omega_n = \sqrt{B'/m''}[(n_1\pi/l_1)^2 + (n_2\pi/l_2)^2]$, where n represents the double subscript (n_1, n_2), l_1 and l_2 represent the lengths of the plate in the x and y directions, B' is the flexural stiffness of the plate and can be expressed by $B' = Eh^3/12(1-\mu^2)$, where E is the modulus of elasticity and μ is the Poisson ratio, and m'' is the mass distribution. φ_n is the nth eigenfunction of the plate given by $\varphi_n(x, y) = \sin(n_1\pi x/l_1)\sin(n_2\pi y/l_2)$, Λ_n is a normalization term given by $\Lambda_n = \frac{1}{4}l_1 l_2 m''$ and η is the hysteretic loss factor which is used to characterise the damping in this case.

Consider the case of a single secondary force controlled by a single primary force. In the mobility formulation the linear velocity v_S induced, for example, by the primary force f_P at the location of the secondary force f_S can be expressed as

$$v_S = M_{SP} f_P. \tag{53}$$

One can easily deduce from equations (52) and (53) that the *linear* mobility is given by

$$M_{SP} = \mathrm{j}\omega \sum_{n=1}^{\infty} \frac{\varphi_n(x_S, y_S)\varphi_n(x_P, y_P)}{\Lambda_n[\omega_n^2(1+\mathrm{j}\eta) - \omega^2]}. \tag{54}$$

It can be noted that $M_{SP} = M_{PS}$, which allows the use of equation (12) in the particular case of the total power minimization strategy, that takes into account only the real part R_{SP} of the mobility M_{SP} and is given by

$$R_{SP} = \mathrm{Re}\{M_{SP}\} = \sum_{n=1}^{\infty} \eta\omega\omega_n^2 \frac{\varphi_n(x_S, y_S)\varphi_n(x_P, y_P)}{\Lambda_n[(\omega_n^2 - \omega^2)^2 + \omega_n^4\eta^2]}. \tag{55}$$

Using the general equation for the optimal set of secondary forces in order to minimise total power when a primary force is applied on the plate, equation (12) in section 2.1, leads to the following expression for the power reduction:

$$W_{T\,min}/W_{PP} = 1 - R_{SP}^2/R_{PP}R_{SS}. \tag{56}$$

The following characteristics for the plate were chosen: the material used is steel, the thickness $h = 1$ mm, the lengths $l_1 = l_2 = 1$ m (hence the plate is square), the damping loss factor $\eta = 0{\cdot}02$, and the first 30×30 modes in equation (5.5) which have natural frequencies up to 4·3 kHz were taken into account. The primary force was assumed to be located on the diagonal of the plate and close to a corner at $(x_P = 0.7, y_P = 0{\cdot}7)$ as shown in Figure 12(a). From equation (55), R_{SS} can be calculated, which in this case is given by

$$R_{SS} = \sum_{n=1}^{\infty} \eta\omega\omega_n^2 \frac{\varphi_n^2(x_S, y_S)}{\Lambda_n[(\omega_n^2 - \omega^2)^2 + \omega_n^4\eta^2]}, \tag{57}$$

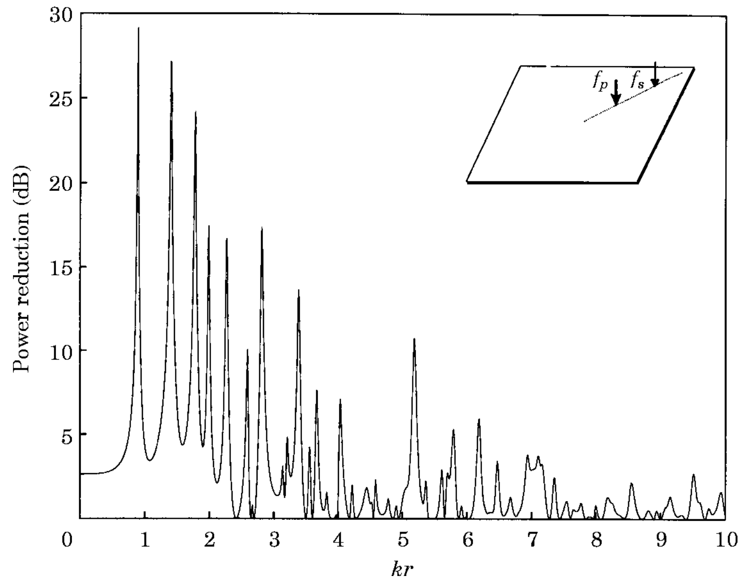

Figure 13. Power reduction against kr in the case of a primary force and a secondary force by using total power minimization on an infinite plate with $x_S = 0{\cdot}84$, $y_S = 0{\cdot}84$.

and R_{PP} is

$$R_{PP} = \sum_{n=1}^{\infty} \eta\omega\omega_n^2 \frac{\varphi_n^2(x_P, y_P)}{\Lambda_n\,[(\omega_n^2 - \omega^2)^2 + \omega_n^4\,\eta^2]}. \tag{58}$$

With equations (55), (57) and (58), the power attenuation given by equation (56) can be calculated: this is plotted in Figure 13 against normalized frequency for a particular location of the secondary force, with the wavenumber $k = (\omega^2 m''/B')^{1/4}$. For the same secondary location one can also plot the total power input to the plate against the

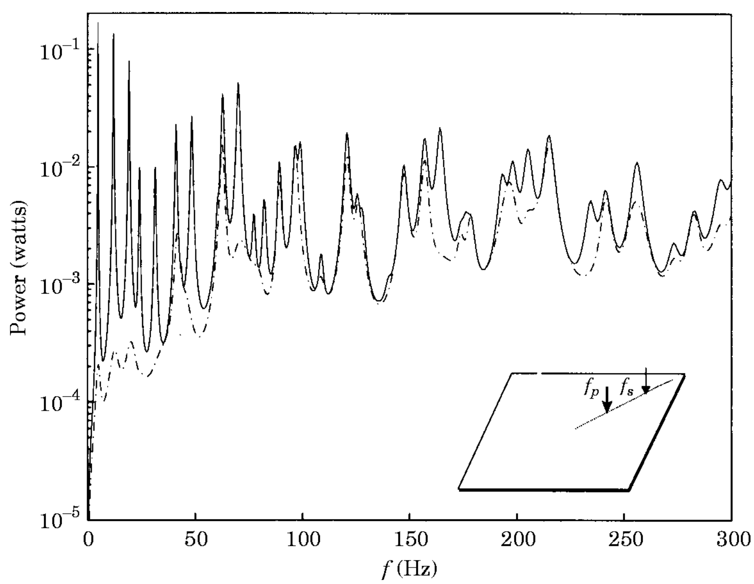

Figure 14. Total power against frequency for a primary force controlled by a secondary force without (——) and with (– · – · –) control by using total power minimization on an finite plate with $x_S = 0{\cdot}84$, $y_S = 0{\cdot}84$.

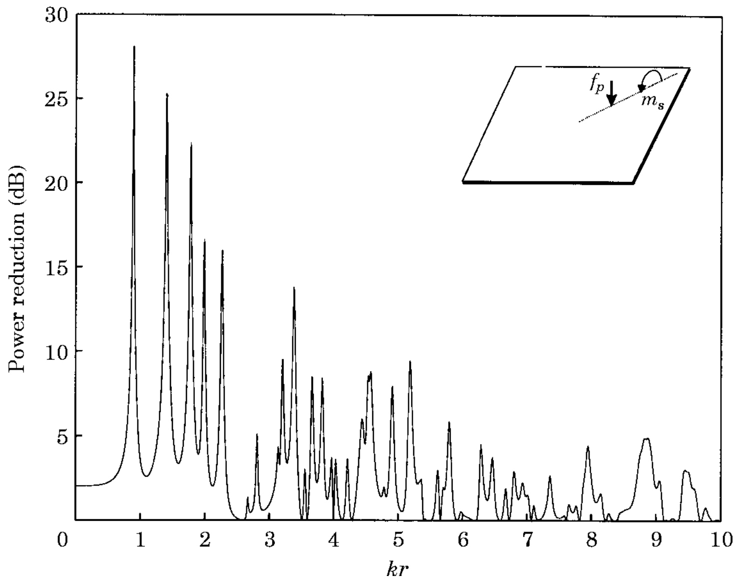

Figure 15. Power reduction against kr for a primary moment force controlled by a secondary moment by using total power minimization on a finite plate with $x_S = 0{\cdot}84$, $y_S = 0{\cdot}84$.

excitation frequency, in Hz, with and without control (see Figure 14), which shows how most modes at the lower frequencies are controlled by the single secondary force, but that other modes, for which the secondary source location lies near a nodal line, are not significantly reduced.

Consider now the case of a single primary force controlled by a single secondary moment. To estimate the available power reduction when using the total power minimization strategy one needs to calculate the effect of a moment on the linear velocity. This can be expressed in terms of the *linear–angular* mobility. In this case, as has been seen in section 2.2, full reciprocity cannot be assumed and equation (23) must be used for the optimal set of secondary forces in order to minimize total power when a set of primary moments is applied on a finite plate. The power reduction can be expressed by analogy with equation (24), with $R'_{PS} = S'_{SP}$ where $R'_{PS} = \mathrm{Re}\,\{M'_{PS}\}$ and $S'_{PS} = \mathrm{Re}\,\{P'_{PS}\}$, as

$$W_{Ta}\,/W_{PP} = 1 - S^2_{PS}\,/S_{SS}\,R_{PP}. \tag{59}$$

In this case R_{PP} is given by equation (58) and the following expressions are derived in Appendix B

$$S_{PS} = \mathrm{Re}\,\{P'_{PS}\} = \sum_{n=1}^{\infty} \eta\omega\omega_n^2 \frac{\varphi_n\,(x_P, y_P)\psi_n^{\delta_S}\,(x_S, y_S)}{\Lambda_n\,[(\omega_n^2 - \omega^2)^2 + \omega_n^4\,\eta^2]}, \tag{60}$$

$$S_{SS} = \mathrm{Re}\,\{P_{SS}] = \sum_{n=1}^{\infty} \eta\omega\omega_n^2 \frac{\psi_n^{\delta_S}\,(x_P, y_P)^2}{\Lambda_n\,[(\omega_n^2 - \omega^2)^2 + \omega_n^4\,\eta^2]}. \tag{61}$$

Here one takes $\delta_S = -135^\circ$ in order to align the secondary moment with the diagonal of the plate and with the secondary force as shown in Figure 12(a). The power reduction is plotted in Figure 15 and the total power before and after control in Figure 16.

If a single primary moment is controlled by a single secondary moment one has to calculate the angular velocity induced by a moment and thus to express the *angular* mobility. One can assume reciprocity in this case and equation (28) in section 2.3 leads to the calculation of the power reduction, which is given by

$$W_{T\,min}/W_{PP} = 1 - S_{SP}^2/S_{PP}\,S_{SS}. \tag{62}$$

Equation (62) can be used to calculate the available power reduction when using the total power minimization strategy, with S_{SS} given by equation (61) and

$$S_{PP} = \mathrm{Re}\,\{P_{PP}\} = \sum_{n=1}^{\infty} \eta\omega\omega_n^2 \frac{\psi_n^{\delta_P}(x_P, y_P)^2}{\Lambda_n\,[(\omega_n^2-\omega^2)^2+\omega_n^4\eta^2]}, \tag{63}$$

$$S_{SP} = \mathrm{Re}\,\{P_{SP}\} = \sum_{n=1}^{\infty} \eta\omega\omega_n^2 \frac{\psi_n^{\delta_S}(x_S, y_S)\psi_n^{\delta_P}(x_P, y_P)}{\Lambda_n\,[(\omega_n^2-\omega^2)^2+\omega_n^4\eta^2]}, \tag{64}$$

which are derived in Appendix B. Here one chooses $\delta_S = -135^\circ$ and $\delta_P = 45^\circ$, to align both of the moments with the diagonal of the plate. The power reduction is plotted in Figure 17 and the total power in Figure 18.

As has been seen in section 2.4, it may be possible to use a collocated secondary force and moment to control, for instance, a single primary force. The main difference here is that now, because of the finite nature of the plate, there are coupling point-mobility terms [25] which link either the linear velocity to a collocated moment excitation ($v_S/m_S = P'_{SS}$) or the angular velocity to a collocated force excitation ($w_S/f_S = M'_{SS}$). Thus, when only the two collocated secondary sources f_S and m_S are exciting the finite plate, the total power input by them is given by $W_{f_S m_S} = (1/2)\,\mathrm{Re}\,(f_S^*\,M_{SS} f_S + f_S^*\,P'_{SS}\,m_S + m_S^*\,M'_{SS} f_S + m_S^*\,P_{SS}\,m_S)$. As a consequence of this, it has been found that the optimal control force f_S and moment m_S which minimize the total power input by the primary f_P and secondary f_S, m_S sources cannot be calculated independently by considering either the primary force f_P controlled only by the secondary force f_S (section 2.1) or the primary force f_P controlled only by the secondary moment m_S (section 2.2).

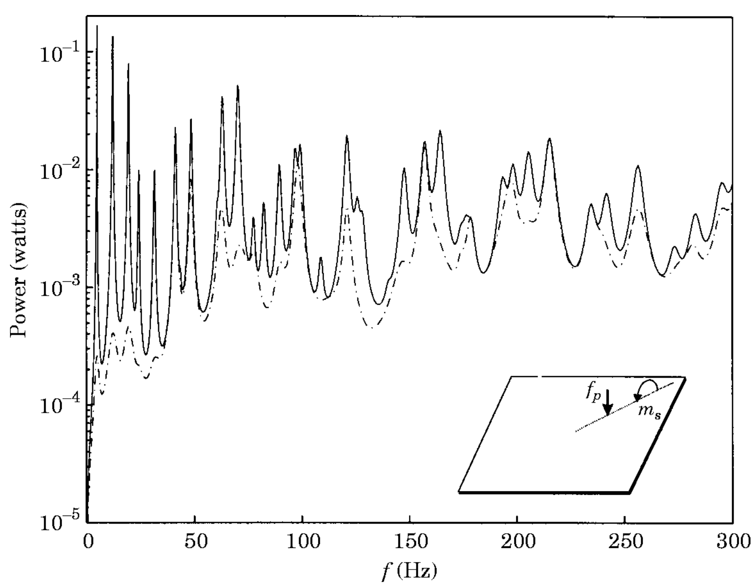

Figure 16. Total power against frequency for a primary force controlled by a secondary moment without (——) and with (– · – · –) control by using total power minimization on a finite plate with $x_S = 0{\cdot}84$, $y_S = 0{\cdot}84$.

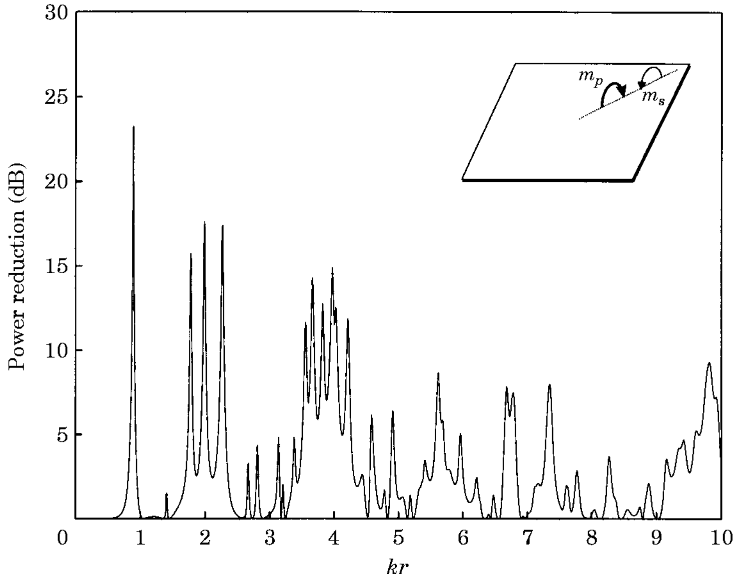

Figure 17. Power reduction against kr for a primary moment controlled by a secondary moment by using total power minimization on a finite plate with $x_S = 0{\cdot}84$, $y_S = 0{\cdot}84$.

Although an analytic expression for the optimal control sources and for the available power reduction when using the total power minimization is not given in this case, because of the complexity, the results of such a calculation are plotted for power reduction in Figure 19 and total power in Figure 20. These results show the improvement in power reduction compared with either that due to the single secondary force or that due to the single secondary moment alone, particularly at higher excitation frequencies.

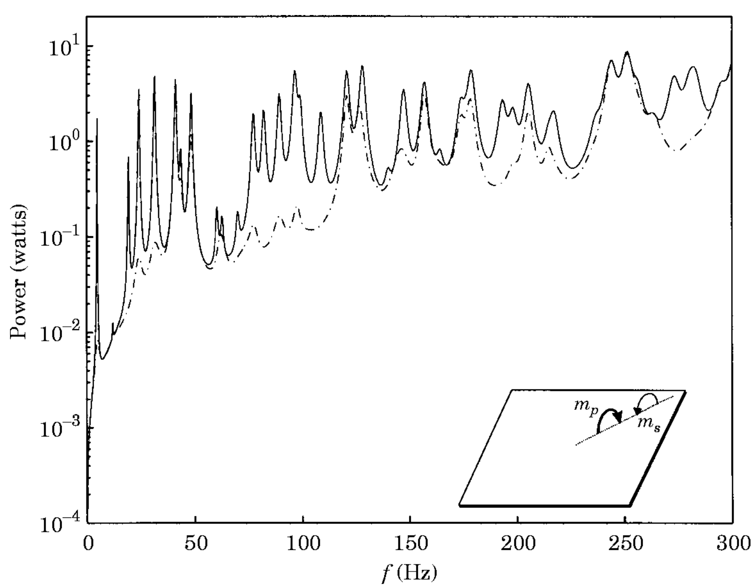

Figure 18. Total power against frequency for a primary moment controlled by a secondary moment without (——) and with (– · – · –) control by using total power minimization on a finite plate with $x_S = 0{\cdot}84$, $y_S = 0{\cdot}84$.

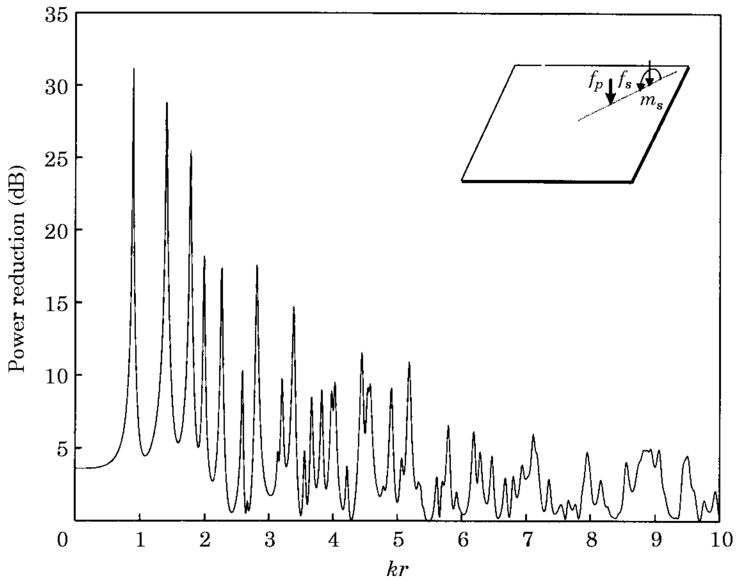

Figure 19. Power reduction against kr for a primary force controlled by a collocated secondary force and moment by using total power minimization on a finite plate with $x_S = 0{\cdot}84$, $y_S = 0{\cdot}84$.

The effects of using several secondary sources arranged in a ring around a primary force have also been considered for the finite plate. Two cases have been analyzed: the first being a set of control forces, while for the second a set of collocated control forces and control moments are used. A uniform distribution of secondary sources around the ring is assumed and the control moments are positioned in vertical planes oriented in the radial direction as is shown in Figure 12(b). The same matrix approach used for the multi-secondary

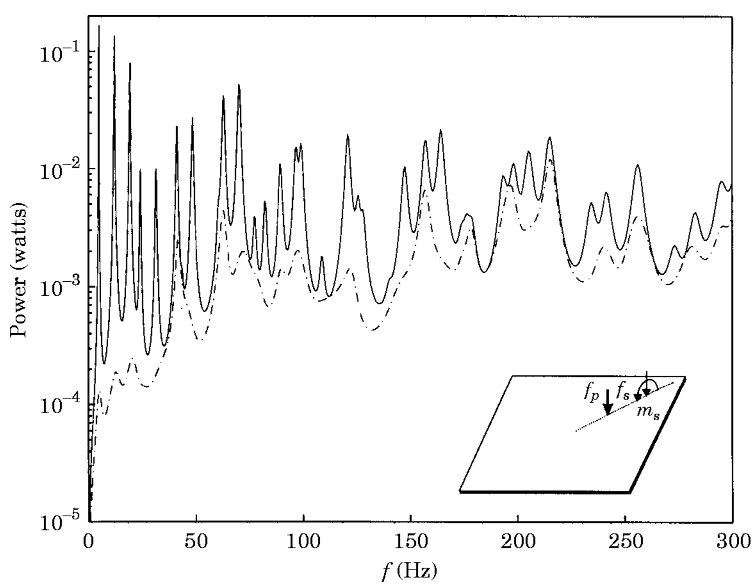

Figure 20. Total power against frequency for a primary force controlled by a collocated secondary force and moment without (——) and with (– · – · –) control by using total power minimization on a finite plate with $x_S = 0{\cdot}84$, $y_S = 0{\cdot}84$.

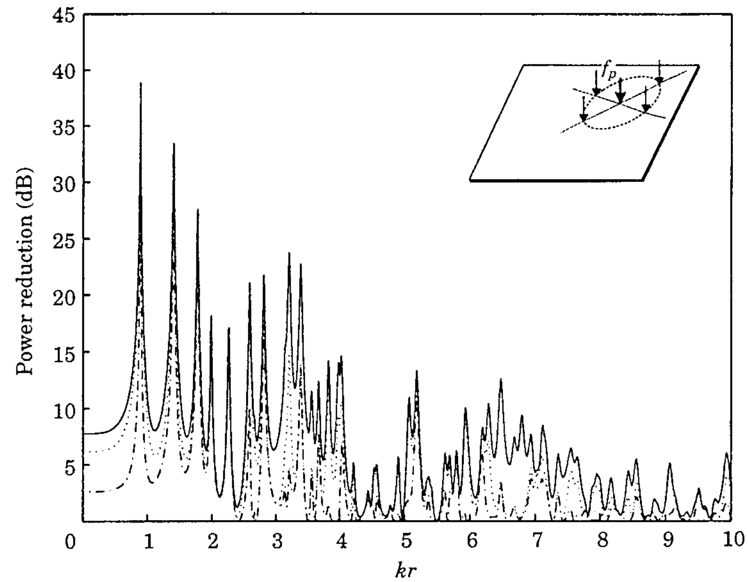

Figure 21. Power reduction against kr for a primary force controlled by one (– · – · –), two (.....) and four (——) secondary forces by using total power minimization on a finite plate with $x_S = 0{\cdot}84$, $y_S = 0{\cdot}84$.

sources control on an infinite plate was used, except that the mobility values derived in Appendix B for the finite plate were used.

In Figure 21 are shown the power reductions, respectively, when one, two or four secondary forces are acting. These graphs show that control is again more effective at resonance, but as the number of control sources is increased the control becomes more effective. For $kr \leqslant 2$ the minimum power reduction achieved with one source is around 4 dB, while when using two or four secondary forces the minimum power reductions are

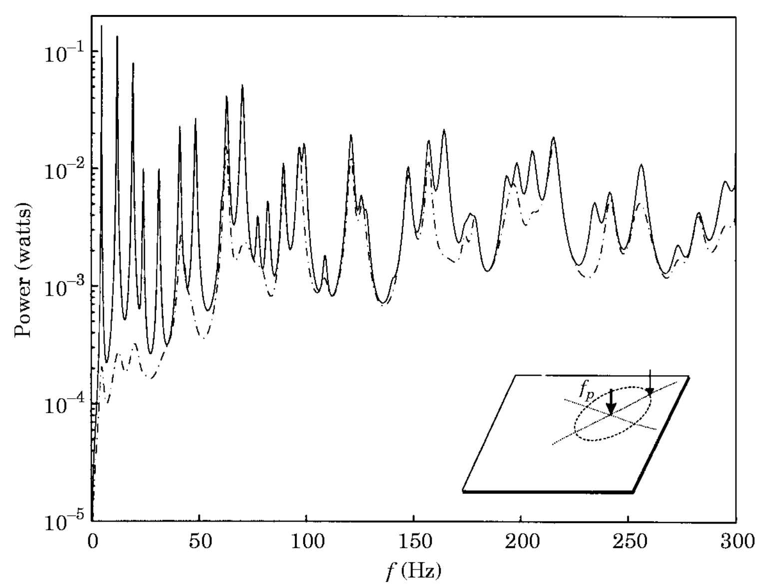

Figure 22. Total power against frequency for a primary force controlled by one secondary force without (——) and with (– · – · –) control by using total power minimization on a finite plate with $x_S = 0{\cdot}84$, $y_S = 0{\cdot}84$.

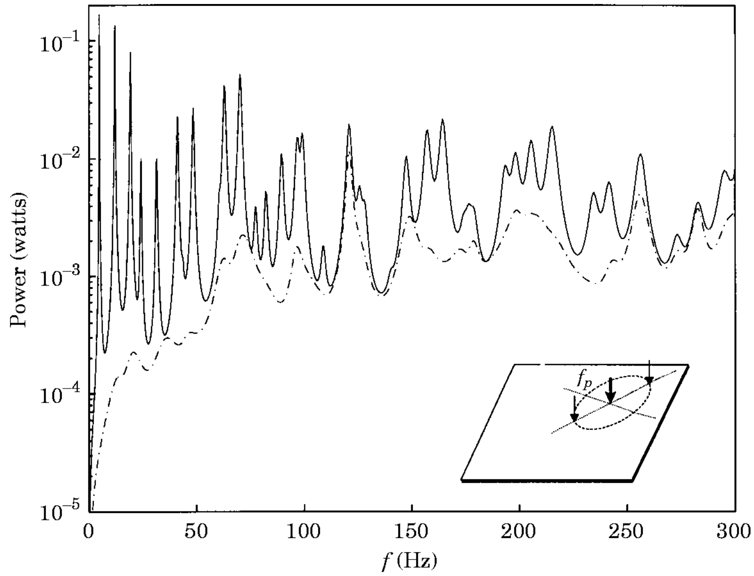

Figure 23. Total power against frequency for a primary force controlled by two secondary forces without (——) and with (– · – · –) control by using total power minimization on a finite plate with $x_S = 0{\cdot}84$, $y_S = 0{\cdot}84$.

respectively around 6 dB and around 7·5 dB. When using one control force the first null in the power reduction curve is for $kr \approx 2{\cdot}4$, while when using two or four control sources the first null is obtained, respectively, for $kr \approx 2{\cdot}6$ and $kr \approx 4{\cdot}9$. This behaviour is rather different from that seen in section 2.5 for the infinite plate and it is due to the asymmetric position of the ring with respect to the edge of the plate. The total power input in the three control cases considered is shown in Figures 22–24.

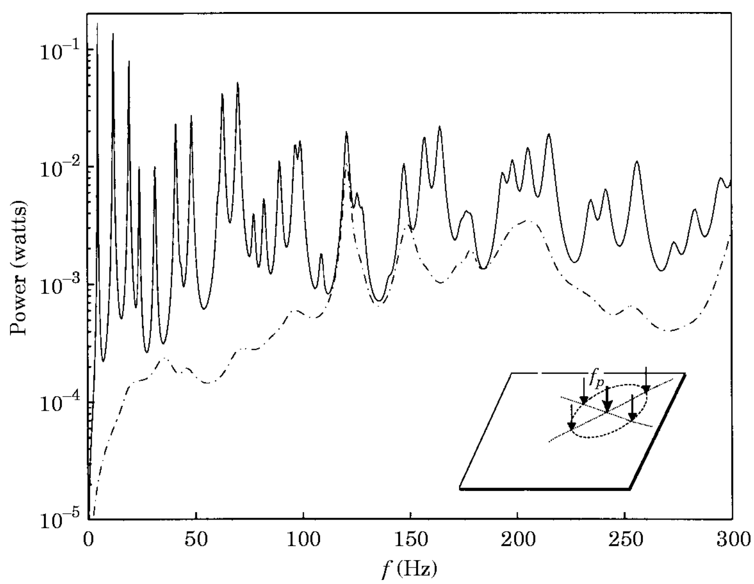

Figure 24. Total power against frequency for a primary force controlled by four secondary forces without (——) and with (– · – · –) control by using total power minimization on a finite plate with $x_S = 0{\cdot}84$, $y_S = 0{\cdot}84$.

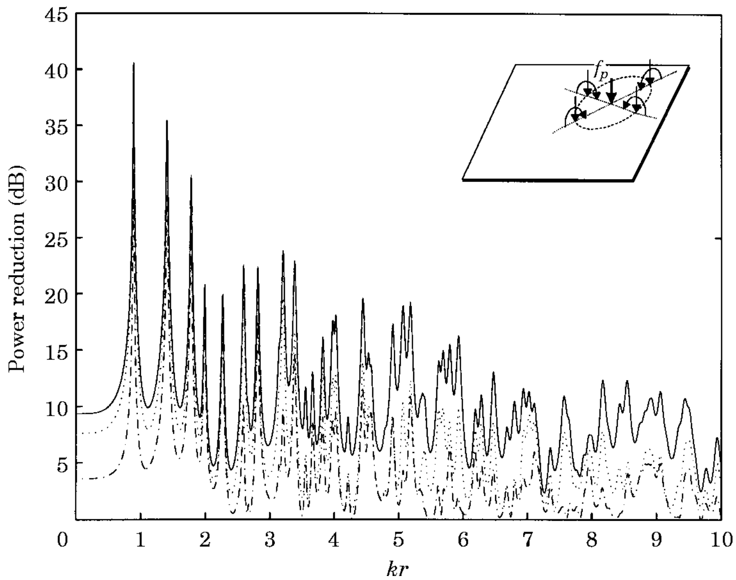

Figure 25. Power reduction against kr for a primary force controlled by one (– · – · –), two (.....) and four (——) collocated secondary forces and moments by using total power minimization on a finite plate with $x_S = 0{\cdot}84$, $y_S = 0{\cdot}84$.

For a ring of collocated secondary forces and moments which are distributed as shown in Figure 12(b), in Figure 25 are shown the power reductions, respectively, when one, two or four pairs of secondary sources are acting. The addition of control moments to the control forces produces a large benefit, since the power reduction with two or four collocated forces and moments never assumes null values for $0 \leqslant kr \leqslant 10$ and the minimum power reductions are, respectively, around 2 dB and around 4·5 dB within this range of kr. Such improvement becomes more clear when considering the total power input into the plate in the three cases, as shown in Figures 26–28. Upon comparing a case with the same number of control sources, for example when the control is obtained by using four secondary forces (Figure 24) and two pair of collocated forces and moments (Figure 27), it is evident that the simultaneous action of the forces and moments gives a more uniform reduction in input power.

4.2. MAXIMIZING SECONDARY SOURCE POWER ABSORPTION

In the case of a single primary force controlled by a single secondary force, one uses equation (36) in section 3.1, which gives the power reduction in the case of maximization of the power absorption of the secondary force. With $\mathbf{M}_{SP} = M_{SP} = R_{SP} + \mathrm{j}X_{SP}$ given by equation (54), $\mathbf{R}_{PP} = R_{PP}$ given by equation (58) and $\mathbf{R}_{SS} = R_{SS}$ given by equation (57), this leads to the following expression for the power reduction when control is applied:

$$W_{Ta}/W_{PP} = 1 - (\tfrac{3}{4} R_{SP}^2 - \tfrac{1}{4} X_{SP}^2)/R_{PP} R_{SS}. \tag{65}$$

The available power reduction given by equation (65) is plotted against (kr) in Figure 29 and the total power against frequency with and without control in Figure 30.

For a single primary force controlled by a single secondary moment, equation (43) in section 3.2 leads to the following expression for the power reduction when using the maximization of the power absorption of the secondary force:

$$W_{Ta}/W_{PP} = 1 - \tfrac{3}{4}(Y_{PS}'^2 - \tfrac{1}{4} S_{PS}'^2)/R_{PP} R_{SS}. \tag{66}$$

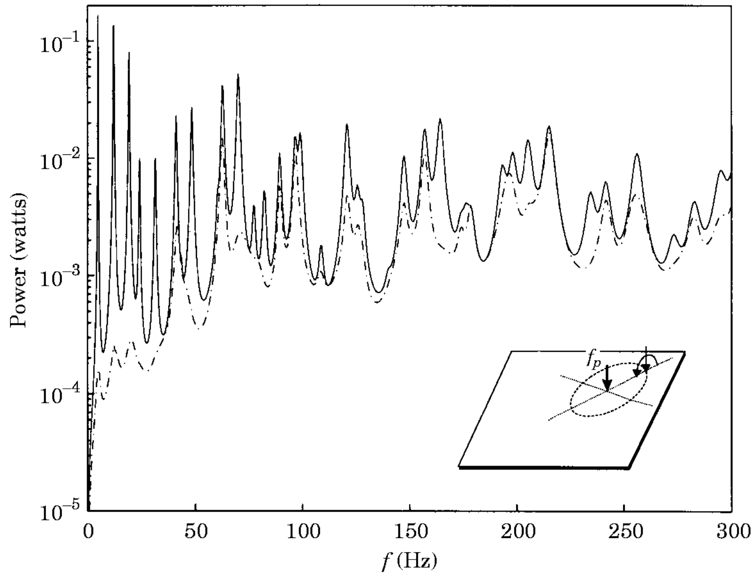

Figure 26. Total power against frequency for a primary force controlled by one collocated secondary force and moment without (——) and with (– · – · –) control by using total power minimization on a finite plate with $x_S = 0{\cdot}84$, $y_S = 0{\cdot}84$.

One knows that $S'_{PS} = R'_{PS}$ due to reciprocity, which is given by equation (60), R_{PP} is given by equation (58), S_{SS} is given by equation (61) and $Y'_{PS} = \mathrm{Im}\,\{P'_{PS}\}$ is derived in Appendix B as

$$Y'_{PS} = \sum_{n=1}^{\infty} \omega(\omega_n^2 - \omega^2)\,\frac{\varphi_n\,(x_P, y_P)\psi_n^{\delta_S}\,(x_S, y_S)}{\Lambda_n\,[(\omega_n^2 - \omega^2)^2 + \omega_n^4\,\eta^2]}. \tag{67}$$

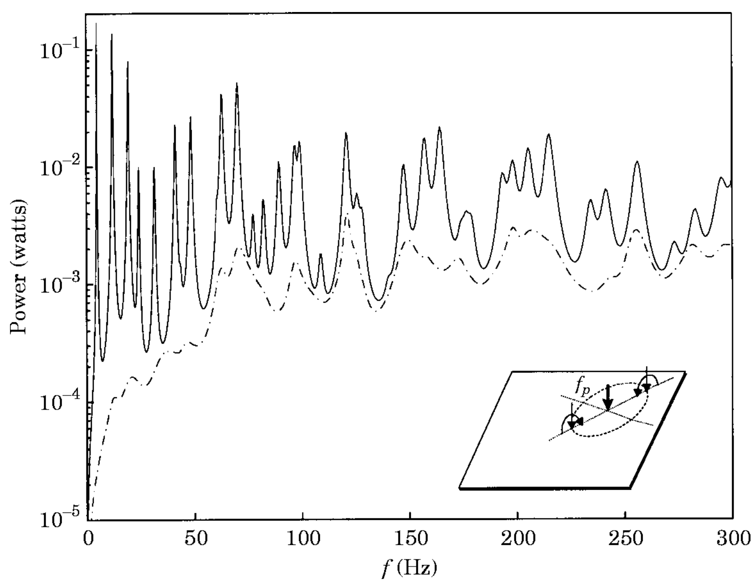

Figure 27. Total power against frequency for a primary force controlled by two collocated secondary forces and moments without (——) and with (– · – · –) control by using total power minimization on a finite plate with $x_S = 0{\cdot}84$, $y_S = 0.84$.

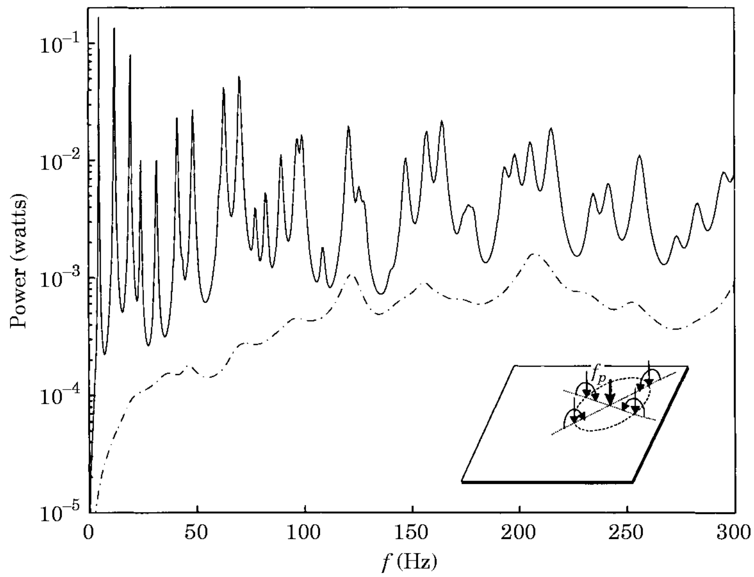

Figure 28. Total power against frequency for a primary force controlled by four collocated secondary forces and moments without (——) and with (– · – · –) control by using total power minimization on a finite plate with $x_S = 0{\cdot}84$, $y_S = 0{\cdot}84$.

The choice $\delta_S = -135^\circ$ is made in order to align the secondary moment on the diagonal of the plate and with the primary force. Thus the available power reduction is given by equation (66) and is plotted against kr in Figure 31, and the total power is plotted against frequency in Figure 32.

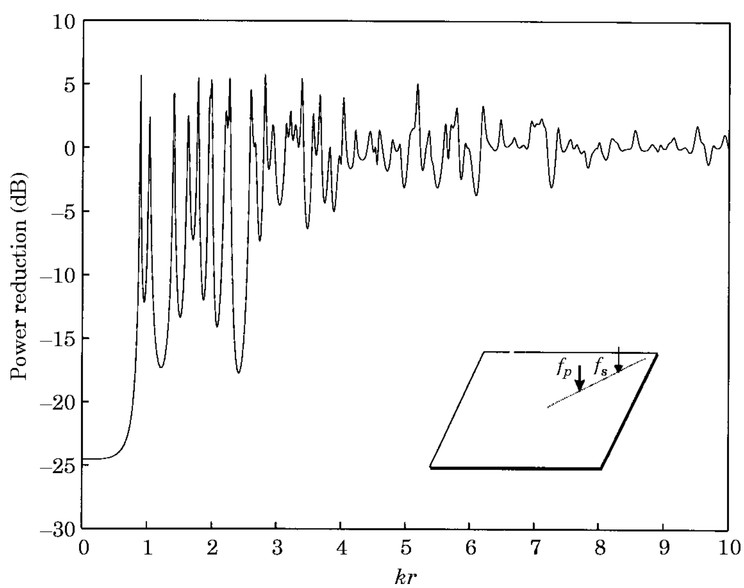

Figure 29. Power reduction against kr for a primary force controlled by a secondary force by using maximization of the secondary source power absorption on a finite plate with $x_S = 0{\cdot}84$, $y_S = 0{\cdot}84$.

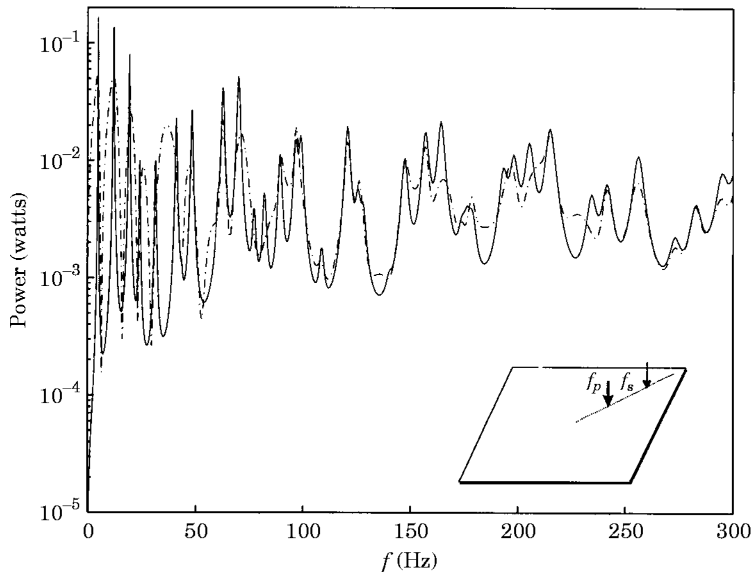

Figure 30. Total power against frequency for a primary force controlled by a secondary force without (——) and with (– · – · –) control by using maximization of the secondary source power absorption on a finite plate with $x_S = 0{\cdot}84$, $y_S = 0{\cdot}84$.

If one now considers the case of a single primary moment controlled by a single secondary moment, equation (50) in section 3.3 allows one to calculate the power reduction in the case of maximizing the power absorption. This leads to the expression

$$W_{Ta}/W_{PP} = 1 - (\tfrac{3}{4}S_{SP}^2 - \tfrac{1}{4}Y_{SP}^2)/S_{PP}S_{SS}. \tag{68}$$

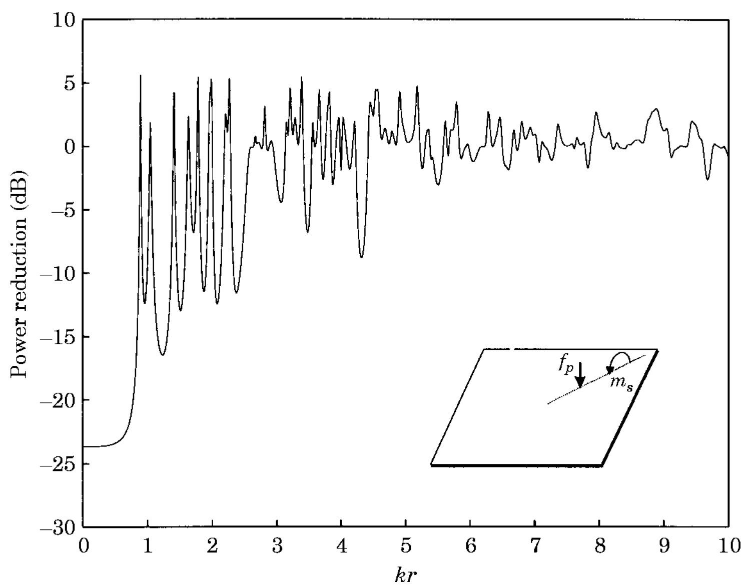

Figure 31. Power reduction against kr for a primary force controlled by a secondary moment by using maximization of the secondary source power absorption on a finite plate with $x_S = 0{\cdot}84$, $y_S = 0{\cdot}84$.

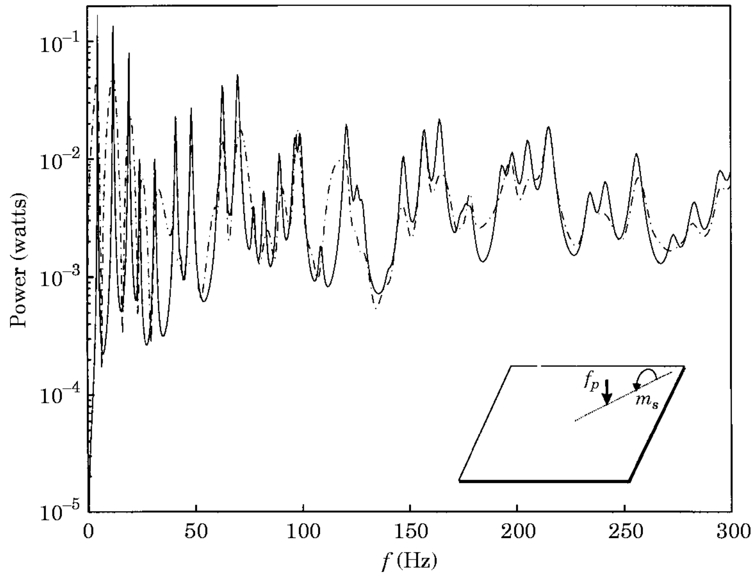

Figure 32. Total power against frequency for a primary force controlled by a secondary moment without (——) and with (– · – · –) control by using maximization of the secondary source power absorption on a finite plate with $x_S = 0{\cdot}84$, $y_S = 0{\cdot}84$.

S_{SP} is given by equation (64), S_{PP} is given by equation (63), S_{SS} is given by equation (61) and $Y_{SP} = \mathrm{Im}\,\{P_{SP}\}$ is derived in Appendix B as

$$Y_{SP} = \sum_{n=1}^{\infty} \omega(\omega_n^2 - \omega^2)\,\frac{\psi_n^{\delta_S}(x_S, y_S)\psi_n^{\delta_P}(x_P, y_P)}{\Lambda_n\,[(\omega_n^2 - \omega^2)^2 + \omega_n^4\,\eta^2]}. \tag{69}$$

Again, one chooses $\delta_S = -135°$ and $\delta_P = 45°$ in order to align both of the moments on the diagonal of the plate. Thus equation (68) leads to the power reduction which is plotted against kr in Figure 33, and the total power against frequency is plotted in Figure 34.

Finally, the use of a collocated secondary force and moment actuator is considered to control a single primary force acting on a finite plate by using the maximization of the power absorption of the secondary actuator. No analytic expression of the power reduction is given here because of the complexity of the solution, but the results of such a calculation are plotted in terms of power reduction in Figure 35 and the total power in Figure 36.

It is clear from these results that a consequence of the finite nature of the plate is to provide a strong coupling effect between the different actuators acting on the plate. Using a control action based on maximizing the power absorption of the secondary source can then lead to a substantial increase in the total power supplied to the plate, especially at low frequency.

5. CONCLUSIONS

A description has been given in this paper of the performance of two possible strategies that can be used to design an active vibration controller: total power minimization and maximization of the power absorption of the secondary source. For both cases, finite and infinite plates have been considered with force and moment actuators. Analytic solutions

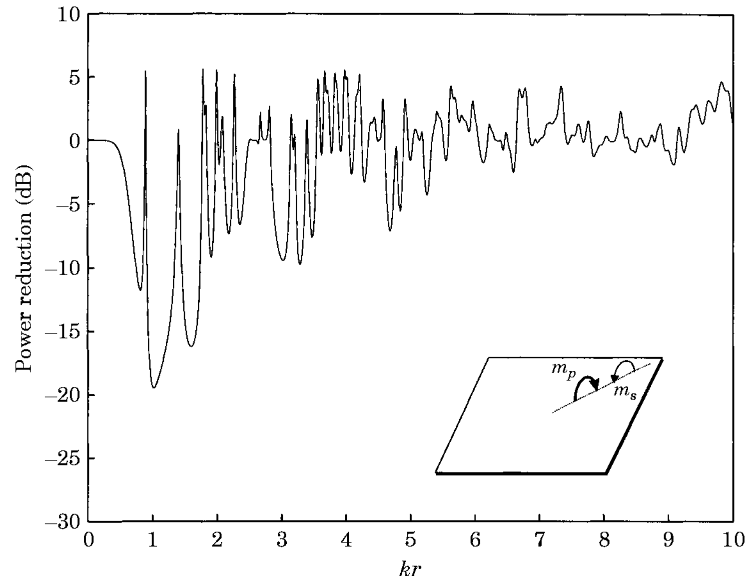

Figure 33. Power reduction against kr for a primary moment controlled by a secondary moment by using secondary source power absorption maximization on a finite plate with $x_S = 0{\cdot}84$, $y_S = 0{\cdot}84$.

for the optimal secondary source and the corresponding power reductions have been derived for each configuration of strategies and actuators.

Substantial reductions (> 10 dB) in the power input to an infinite plate due to a primary force can be obtained with secondary force adjusted to minimize total power input provided that it is placed closer than about one-eighth of a flexural wavelength. A secondary moment can also give reductions of up to about 5 dB in power input, but in

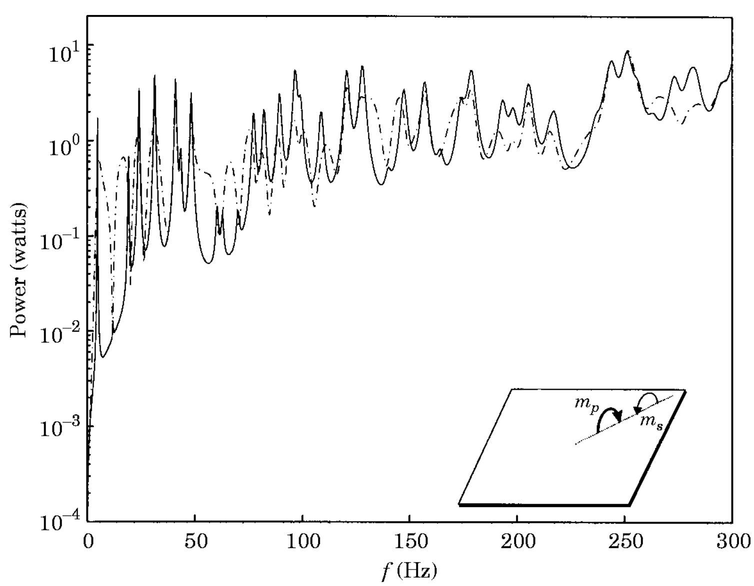

Figure 34. Total power against frequency for a primary moment controlled by a secondary moment without (——) and with (– · – · –) control by using maximization of the secondary source power absorption on a finite plate with $x_S = 0{\cdot}84$, $y_S = 0{\cdot}84$.

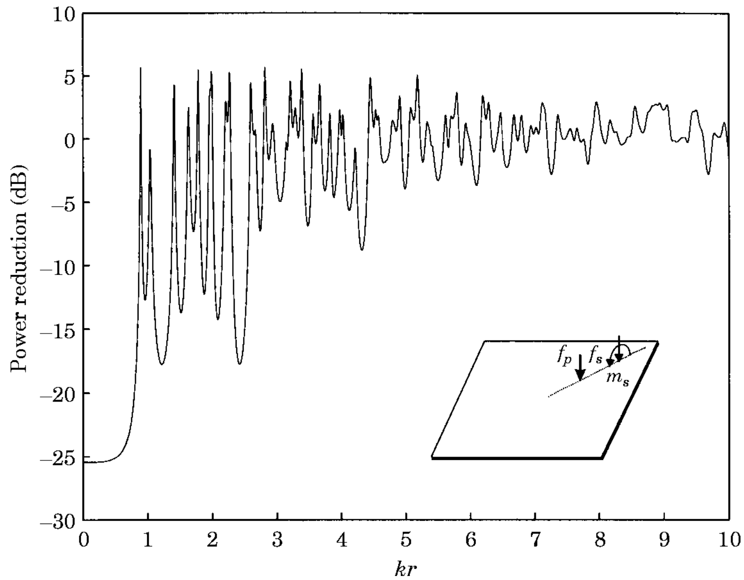

Figure 35. Power reduction against kr for a primary force controlled by a collocated secondary force and moment by using maximization of the secondary source power absorption on a finite plate with $x_S = 0{\cdot}84$, $y_S = 0{\cdot}84$.

this case it must be placed about one-third of a flexural wavelength from the primary force. A combined secondary force and moment acting at the same point on an infinite plate but independently adjusted are able to give reductions, which are for any value of kr, greater than or equal to those obtained either for a single control force or for a single control moment.

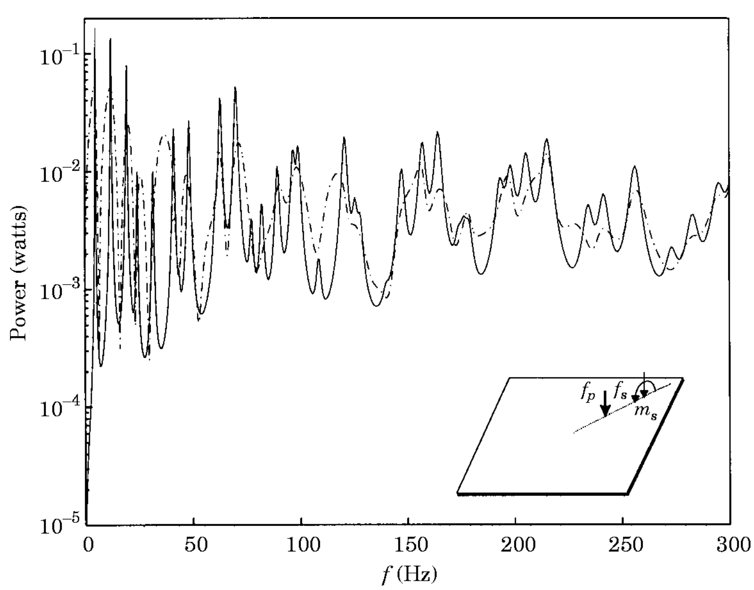

Figure 36. Total power against frequency for a primary force controlled by a collocated secondary force and moment without (——) and with (–·–·–) control by using maximization of the secondary source power absorption on a finite plate with $x_S = 0{\cdot}84$, $y_S = 0{\cdot}84$.

The use of multiple-secondary forces, or collocated forces and moments, arranged in a ring around the primary source can give even better control than a single force or moment or a collocated force and moment. In general, for an equal number of control sources, the best result is achieved by using collocated forces and moments.

Maximization of the power absorption of the secondary source also generally reduces the total power input from both primary and secondary sources on an infinite plate. Reductions in total power of 6 dB can be obtained with a closely spaced secondary force when controlling a primary force on an infinite plate by using this strategy. A secondary moment also gives about 3 dB reduction in the total power input to the infinite plate when placed about a third of a wavelength from the primary force. The large increases in total power output of the sources observed with closely spaced acoustic monopoles when using this control strategy does not appear to occur, except when a secondary moment is positioned very close to a primary moment.

In the case of the finite plate considered here, reductions in total power input of up to 20 dB can be obtained with a single secondary force or moment adjusted to total minimized input power, but only at the natural frequencies of modes with which these sources can efficiently couple. Although for a finite panel the fractional reductions with both a secondary force and moment are no longer equal to the sum of the fractional reductions which could be obtained individually, the action of the two secondary sources is still complementary. At higher frequencies (in particular, for $kr = 5$–10) the two collocated secondary sources work together to achieve reductions of at least 2 dB and give about 8 dB on average, whereas the force or moment individually cannot achieve any significant reductions at a number of frequencies one obtains an average reduction of only about 4 dB. The use of several control forces or several collocated control forces and moments give a better performance for the finite plate than that obtained by using a single control force or moment or a collocated force and moment.

The control strategy of maximizing the power absorption of the secondary sources on the finite plate can result in increases of up to 20 dB in the total power supplied to the plate, particularly at low frequencies. This is true for both force and moment sources and reflects the increased coupling which exists on a finite plate, not just between forces and linear velocities, but also between moments and linear velocities. Similar results were reported by Brennan *et al*. [22] for a finite beam.

As long as it is possible to have knowledge of all the power transmission paths, the total power minimization strategy thus offers better results than maximizing the power absorption of the secondary source. Although the strategy of maximizing the power absorbed by the secondary sources should be avoided on structures with strong reflections, it may be worthwhile on more anechoic structures and may be considerably simpler to implement than total power minimization. Combining a force and moment in the design of the secondary source appears to improve significantly the action of the controller, particularly on a finite structure.

ACKNOWLEDGMENT

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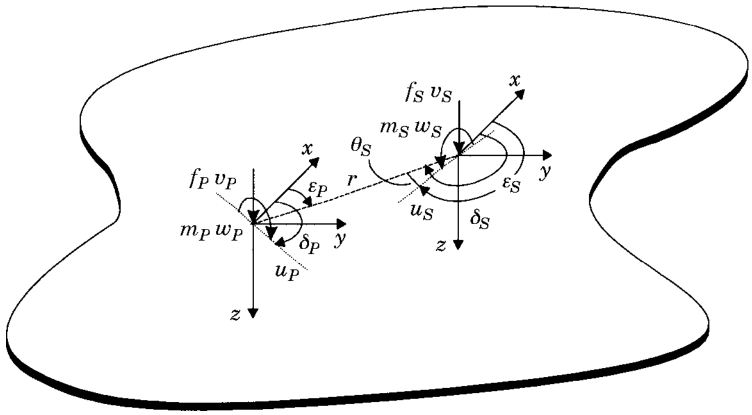

Figure A1. The notation used for the force and moment and for the linear and angular velocities in an infinite plate.

APPENDIX A: MOBILITY TERMS FOR AN INFINITE PLATE

In this appendix the analytic form of the terms of the mobility matrix are derived for an infinite plate when one or both of the two main sources (primary source and secondary source) are made up by forces or moments. The case of two force sources has already been carried out by Jenkins *et al*. [20].

Considered here is an infinite plate excited first by a single primary force f_P and controlled by a single secondary moment m_S. One wants to find the analytic expression for the optimal solution of the secondary force in order to minimize, for example, the total power output, which is given by

$$W_T = \tfrac{1}{2}\,\mathrm{Re}\left\{m_S^*\, P_{SS}\, m_S + m_S^*\, M'_{SP} f_P + f_P^*\, P'_{PS}\, m_S + f_P^*\, M_{PP} f_P\right\} \tag{A1}$$

For this purpose one needs to calculate: first, the angular velocity induced by the primary force at the location of the secondary moment (represented by P'_{SP}, f_P); second, the linear velocity induced by the secondary moment at the location of the primary force (represented by $M'_{SP} f_P$); and, finally, the angular velocity induced by the secondary moment at its own location (represented by $P_{SS}\, m_S$). The mobility P_{SS} relating a moment m_S with the angular velocity at the same point w_S has been derived from a general case in such a way to show how to calculate the angular velocity at the location of the primary source (represented by $P_{PS}\, m_S$). In this way the formulation presented in this appendix gives the analytic expression of the four transfer and input mobilities that are needed to study the cases having a set of primary and secondary sources composed by both force and moment excitations.

In Figure A1 is shown the notation used for the forces and moments and for the linear and angular velocities at the primary source and secondary source positions. At each of these two points a local system of reference composed of a right-handed triple of vectors (x, y, z) is defined. The positive forces (f_P and f_S) and linear velocities (v_P and v_S) are oriented in the z direction, while the positive moments (m_P and m_S) and angular velocities (w_P and w_S) positioned in a vertical plane having a general orientation are determined by the angles δ_P and δ_S that are positive with reference to the right-handed screw rule as shown in Figure A1. The formulation of the transfer mobility P_{PS} requires a second set of angles ε_P and ε_S which are defined as the angles between the x axes and the segment joining the primary source and the secondary source positions. Also these angles are defined as positive with reference to the right-handed screw rule. This notation has been introduced for the particular case of a primary source and a secondary source, but can be assumed in general also to find the relations between excitations and velocities at any pair of points at which primary or secondary sources act.

A.1. ANALYTIC EXPRESSION FOR P'_{PS} AND P'_{SP}

In order to create the moment effect m_S, one applies two similar forces with opposite phases $+f$ and $-f$ a length $2e$ away and apart from the point P along the u_S line; then one reduces the distance between the two forces to zero. It is known that the *complex* velocity v induced at a given location on an infinite plate by a force f is given by [25]

$$v = f\beta_0\,[\mathrm{H}_0^{(2)}(kr) - \mathrm{H}_0^{(2)}(-\mathrm{j}kr)], \tag{A2}$$

where r is the distance between the primary and the secondary sources, $\mathrm{H}_0^{(2)}()$ is the second Hankel function of zeroth order and $\beta_0 = \omega/8Bk^2$. Moreover, for a real argument, $\mathrm{H}_0^{(2)}(kr) = \mathrm{J}_0(kr) - \mathrm{j}\mathrm{Y}_0(kr)$, where $\mathrm{J}_0()$ is the first kind of Bessel function of order zero and $\mathrm{Y}_0()$ is the second kind of Bessel function of order zero. For an imaginary argument one can write $\mathrm{H}_0^{(2)}()$ in the form $\mathrm{H}_0^{(2)}(-\mathrm{j}kr) = \mathrm{j}(2/\pi)\mathrm{K}_0(kr)$, where $\mathrm{K}_0()$ is the modified second kind of Bessel function of order zero. Thus one can express the complex velocity v by

$$v = f\beta_0\,[\mathrm{J}_0(kr) - \mathrm{j}(\mathrm{Y}_0(kr) + (2/\pi)\mathrm{K}_0(kr))] = fM(kr), \tag{A3}$$

where M is the *linear* mobility and depends on kr. If one calls v_P^+ the complex linear velocity generated at the location P by the point force $+f_S$ and v_P^- the complex linear velocity generated at the same location by the point force $-f_S$, with reference to the notation shown in Figure A2, one can express the total complex linear velocity V_P at P as

$$V_P = v_P^+ + v_P^- = f_S\,[M(k(r - e\cos\theta_S)) - M(k(r + e\cos\theta_S))], \tag{A4}$$

where $\theta_S = \varepsilon_S - \delta_S$. Let now $a \to 0$ to create the moment effect and then calculate the total complex velocity v_P by taking the limit of V_P, i.e., $v_P = \lim_{e\to 0} V_P$, which leads to

$$v_P = -f_S\left(\lim_{e\to 0}\,[M(k(r + e\cos\theta_S)) - M(k(r - e\cos\theta_S))]\right)$$

$$= -m_S\cos\theta_S\left(\lim_{e\to 0}\frac{M(k(r + e\cos\theta_S)) - M(k(r - e\cos\theta_S))}{2e\cos\theta_S}\right)$$

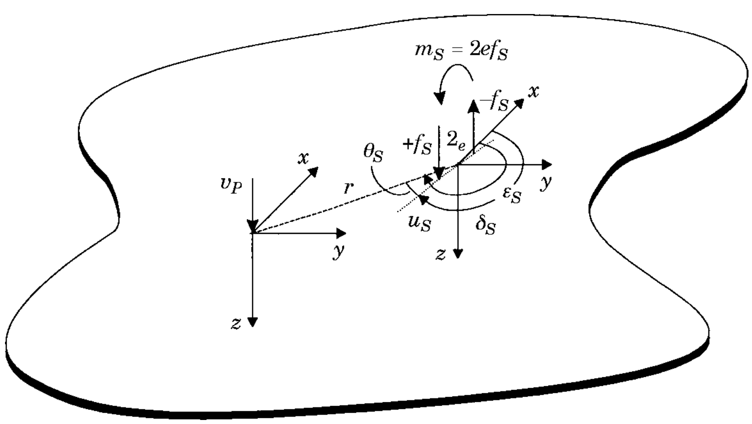

Figure A2. The linear velocity at position P induced by a moment aligned with the segment u_S and applied at position S.

$$-m_S \cos\theta_S \left(\lim_{\alpha\to 0} \frac{M(k(r+\alpha)) - M(k(r-\alpha))}{2\alpha}\right), \tag{A5}$$

where $m_S = 2ef_S$ is the moment and $\alpha = e\cos\theta_S$. The term in brackets is simply the expression for the derivative of M with respect to r. Hence the expression for the linear velocity is

$$v_P = -m_S \cos\theta_S\, \partial M/\partial r. \tag{A6}$$

With $M = \beta_0\,[\mathrm{J}_0(kr) - \mathrm{j}(\mathrm{Y}_0(kr) + (2/\pi)\mathrm{K}_0(kr)]$, equation (A6) becomes

$$v_P = k\beta_0 \cos\theta_S\,[\mathrm{J}_1(kr) - \mathrm{j}(\mathrm{Y}_1(kr) + (2/\pi)\mathrm{K}_1(kr)]m_S. \tag{A7}$$

One can then deduce from equation (A7) that the *angular–linear* mobility P'_{PS} is given by

$$P'_{PS} = k\beta_0 \cos\theta_S\,[\mathrm{J}_1(kr) - \mathrm{j}(\mathrm{Y}_1(kr) + (2/\pi)\mathrm{K}_1(kr))]. \tag{A8}$$

An equation of the same type as equation (A7) is obtained when the linear velocity at the secondary source v_S is due to a primary moment m_P and then the equation of the *angular*-*linear* mobility P'_{SP} is given by

$$P'_{SP} = k\beta_0 \cos\theta_P\,[\mathrm{J}_1(kr) - \mathrm{j}(\mathrm{Y}_1(kr) + (2/\pi)\mathrm{K}_1(kr))], \tag{A9}$$

where $\theta_P = \varepsilon_P - \delta_P$. The formulation presented here for P'_{PS} and P'_{SP} is valid for any pair of positions on which act moments and forces that interfere.

A.2 ANALYTIC EXPRESSION FOR M'_{SP} AND M'_{PS}

The angular velocity w_S induced by the primary force f_P at the location of the secondary moment m_S is now calculated. The angular velocity w_S is related to the linear velocity v_S by

$$w_S = \mathrm{d}v_S/\mathrm{d}u_S = -\cos\theta_S\,\partial v_S/\partial r = -f_P\cos\theta_S\,\partial M/\partial r, \tag{A10}$$

where $\theta_S = \varepsilon_S - \delta_S$. Using equation (A3) for v_S in equation (A10) yields

$$w_S = k\beta_0 \cos\theta_S\,[\mathrm{J}_1(kr) - \mathrm{j}(\mathrm{Y}_1(kr) + (2/\pi)\mathrm{K}_1(kr))]f_P. \tag{A11}$$

Then from equation (A11) one deduces that the *linear–angular* mobility M'_{SP} is given by

$$M'_{SP} = k\beta_0 \cos\theta_S\,[\mathrm{J}_1(kr) - \mathrm{j}(\mathrm{Y}_1(kr) + (2/\pi)\mathrm{K}_1(kr))]. \tag{A12}$$

It can be noted from equations (A8) and (A12) that the effect of the moment and the force on each other is exactly the same.

The angular velocity at the primary source w_P due to a secondary force f_S is given by an equation of the same type as equation (A11), so the equation for the *linear*-*angular* mobility M'_{PS} is

$$M'_{PS} = k\beta_0 \cos\theta_P\,[\mathrm{J}_1(kr) - \mathrm{j}(\mathrm{Y}_1(kr) + (2/\pi)\mathrm{K}_1(kr)], \tag{A13}$$

where $\theta_P = \varepsilon_P - \delta_P$. The formulation presented here for M'_{SP} and M'_{PS} is valid for any pair of positions on which act moments and forces that interfere.

A.3. ANALYTIC EXPRESSIONS FOR P_{SS}, P_{PP}, P_{PS} AND P_{SP}

The *angular* mobility P_{SS} involves the angular velocity induced at the secondary source by the secondary moment itself. However it can be useful to calculate the angular velocity induced at any point on the plate, by a moment when one is dealing with several primary or (and) secondary moments. Thus, for example, the angular velocity at the primary source due to the secondary moment is determined. By using equation (A7) and (A10), the angular velocity at the primary source position w_P induced by the secondary moment m_S is found to be

$$w_P = \{\cos\psi_P \quad \sin\psi_P\}\begin{bmatrix}\cos\theta_S & -(1/r)\sin\theta_S \\ \sin\theta_S & (1/r)\cos\theta_S\end{bmatrix}\begin{Bmatrix}\partial P'_{PS}/\partial r \\ \partial P'_{PS}/\partial\theta_S\end{Bmatrix} m_S, \tag{A14}$$

where $\psi_P = \delta_S - \delta_P$ and $\theta_S = \varepsilon_S - \delta_S$. Therefore, the *angular* mobility P_{PS} is given by

$$P_{PS} = \{\cos\psi_P \quad \sin\psi_P\}\begin{bmatrix}\cos\theta_S & -(1/r)\sin\theta_S \\ \sin\theta_S & (1/r)\cos\theta_S\end{bmatrix}\begin{Bmatrix}\partial P'_{PS}/\partial r \\ \partial P'_{PS}/\partial\theta_S\end{Bmatrix}. \tag{A15}$$

In the particular case of secondary moment m_S and primary angular velocity w_P being aligned, equation (A15) assumes the form

$$P_{PS} = 2\gamma_0\left[\mathrm{J}_0(kr) - \frac{1}{kr}\mathrm{J}_1(kr) - \mathrm{j}\left(\mathrm{Y}_0(kr) - \frac{1}{kr}\mathrm{Y}_1(kr) - \frac{2}{\pi}\left(\mathrm{K}_0(kr) + \frac{1}{kr}\mathrm{K}_1(kr)\right)\right)\right], \tag{A16}$$

where $\gamma_0 = \omega/16B = (k^2/2)\beta_0$. The angular velocity at the secondary source w_S due to a primary moment m_P is given by an equation of the same type as equation (A16), so the equation of the *angular* mobility P_{SP} is given by

$$P_{SP} = \{\cos\psi_S \quad \sin\psi_S\}\begin{bmatrix}\cos\theta_P & -(1/r)\sin\theta_P \\ \sin\theta_P & (1/r)\cos\theta_P\end{bmatrix}\begin{Bmatrix}\partial P'_{SP}/\partial r \\ \partial P'_{SP}/\partial\theta_P\end{Bmatrix}, \tag{A17}$$

where $\psi_s = \delta_P - \delta_S$ and $\theta_P = \varepsilon_P - \delta_P$. With reference to the simplified equations (A16) relating aligned angular velocity and moment that are valid also for the mobility P_{SP} one can deduce that $P_{PP} = P_{SS} = \gamma_0$.

Also in this case, the formulation presented for P_{PS} and P_{SP} is valid for any pair of positions on which act moments and forces that interfere.

APPENDIX B: MOBILITY TERMS FOR A FINITE PLATE

Returning to equation (A1), one wants to calculate the different mobility terms but for the case of a finite plate. Figure B1 shows the notation used for the forces and moments and for the linear and angular velocities at the primary source and secondary source positions. At each of these two points a local system of reference composed of a right-handed triple of vectors (x_P, y_P, z_P) and (x_S, y_S, z_S) is defined. The positive forces (f_P and f_S) and linear velocities (v_P and v_S) are oriented in the z_1 direction, while the positive moments (m_P and m_S) and angular velocities (w_P and w_S) positioned in a vertical plane having a general orientation are determined by the angles δ_P and δ_S that are positive with reference to the right-handed screw rule, as shown in figure A2. The formulation of the mobility terms for a finite plate is a little different from that for the infinite case because

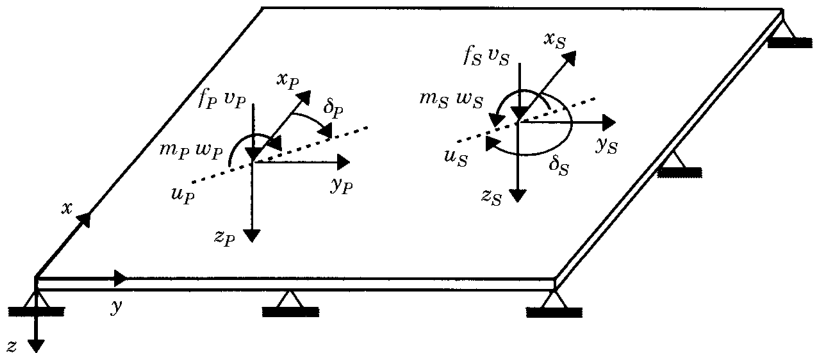

Figure B1. The notation used for the force and moment and for the linear and angular velocities in a finite plate.

it depends on the location of the excitation. For this reason a main system of reference composed of a right-handed triple of vectors (x, y, z) has been fixed at the bottom left-hand corner of the plate, as shown in Figure A2. Another important difference between the infinite plate mobilities and the finite plate mobilities is due to the fact that an input force or moment acting on a finite plate produces, respectively, an angular and a linear velocity at the same point where the force or moment is located. Therefore, the mobility terms M'_{PS}, M'_{SP} or P'_{PS}, P'_{SP} for a finite plate include not only a transfer mobility term but a point mobility term as well. Such point mobility terms are indicated by M'_{PP}, M'_{SS} or P'_{PP}, P'_{SS}.

B.1 ANALYTIC EXPRESSIONS FOR P'_{PS}, P'_{SP}, P'_{PP} AND P'_{SS}

The process is exactly the same as in the infinite case: i.e., as shown in Figure A2. One applies on the plate two opposite forces a length $2r$ away which is reduced to zero in order to create the secondary moment m_S. A primary force f_P is applied at the location (x_P, y_P) on the plate in order to control m_S. This allows one to calculate of the *angular–linear* and *linear–angular* mobility terms as defined by equation (A1), P'_{PS} and M'_{SP}, respectively, which represent, respectively, the linear velocity v_P due to a unit moment m_S and the angular velocity w_S due to a unit force f_P.

One knows that the linear velocity v_P induced at the location (x_P, y_P) by a force f_S located at (x_S, y_S) can be written as

$$v_P = f_S \sum_{n=1}^{\infty} (\eta\omega\omega_n^2 + \mathrm{j}\omega(\omega_n^2 - \omega^2)) \frac{\varphi_n(x_P, y_P)\varphi_n(x_S, y_S)}{\Lambda_n[(\omega_n^2 - \omega^2)^2 + \omega_n^4\eta^2]}. \tag{B1}$$

Hence the total velocity V_P, at the location (x_P, y_P) induced by two opposite forces $+f_S$ (which induces v_P^+) and $-f_S$ (which induces v_P^-) each a distance e away from (x_S, y_S) along the u_S line, is

$$\begin{aligned} V_P &= v_P^+ + v_P^- \\ &= f_S \sum_{n=1}^{\infty} \frac{(\eta\omega\omega_n^2 + \mathrm{j}\omega(\omega_n^2 - \omega^2))}{\Lambda_n[(\omega_n^2 - \omega^2)^2 + \omega_n^4\eta^2]} \varphi_n(x_P, y_P)[\varphi_n(x_S - a, y_S - b) - \varphi_n(x_S + a, y_S + b)], \end{aligned} \tag{B.2}$$

with $\varphi_n(x, y) = \sin(n_1 \pi x/l_1) \sin(n_2 \pi y/l_2)$, $a = e \cos \delta_S$ and $b = e \sin \delta_S$. One knows that the secondary moment $m_S = 2rf_S$ and so one lets $e \to 0$ in equation (B2). After some manipulations the following expression is obtained for the linear velocity v_P at (x_P, y_P):

$$v_P = \lim_{e \to 0} V_P = m_s \sum_{n=1}^{\infty} (\eta\omega\omega_n^2 + \mathrm{j}\omega(\omega_n^2 - \omega^2)) \frac{\varphi_n(x_P, y_P)\psi_n^{\delta_S}(x_S, y_S)}{\Lambda_n[(\omega_n^2 - \omega^2)^2 + \omega_n^4 \eta^2]}. \tag{B3}$$

Here

$$\psi_n^{\delta_S}(x, y) = \frac{n_1 \pi}{l_1} \cos \delta_S \cos\left(\frac{n_1 \pi x}{l_1}\right) \sin\left(\frac{n_2 \pi y}{l_2}\right) + \frac{n_2 \pi}{l_2} \sin \delta_S \sin\left(\frac{n_1 \pi x}{l_1}\right) \cos\left(\frac{n_2 \pi y}{l_2}\right). \tag{B.4}$$

The linear velocity v_P induced at (x_P, y_P) by a moment m_S located at (x_S, y_S) is known to be $v_P = P'_{PS} m_S$. From equation (B3) one deduces the analytic expression for P'_{PS}:

$$P'_{PS} = \sum_{n=1}^{\infty} (\eta\omega\omega_n^2 + \mathrm{j}\omega(\omega_n^2 - \omega^2)) \frac{\varphi_n(x_P, y_P)\psi_n^{\delta_S}(x_S, y_S)}{\Lambda_n[(\omega_n^2 - \omega^2)^2 + \omega_n^4 \eta^2]}. \tag{B5}$$

The same procedure can be adopted to find the mobility term for the linear velocity v_S induced at (x_S, y_S) by a moment m_P located at (x_P, y_P):

$$P'_{SP} = \sum_{n=1}^{\infty} (\eta\omega\omega_n^2 + \mathrm{j}\omega(\omega_n^2 - \omega^2)) \frac{\varphi_n(x_S, y_S)\psi_n^{\delta_P}(x_P, y_P)}{\Lambda_n[(\omega_n^2 - \omega^2)^2 + \omega_n^4 \eta^2]}. \tag{B6}$$

It can be noted that $P'_{SP} \neq P'_{PS}$. The analytic expression for P'_{PP} or P'_{SS} is exactly the same as in equations (B5) or (B6). The mobility P'_{PP} will refer to $\varphi_n(x_P, y_P)$ and $\psi_n^{\delta_P}(x_P, y_P)$ while the mobility P'_{SS} will refer to $\varphi_n(x_S, y_S)$ and $\psi_n^{\delta_S}(x_S, y_S)$.

The formulation presented here for P'_{PS} and P'_{SP} is valid for any pair of positions on which act moments and forces that interfere.

B.2 ANALYTIC EXPRESSIONS FOR M'_{SP}, M'_{PS}, M'_{SS} AND M'_{PP}

One now wants to express now the *linear–angular* mobility M'_{SP} for the angular velocity induced at (x_S, y_S) by the primary force f_P located at (x_P, y_P). The angular velocity w_s can be calculated from the linear velocity v_s by differentiating the linear velocity at the location (x_S, y_S) with reference to the direction given by the line u_s, as shown in Figure B1:

$$w_s = \mathrm{d}v_s / \mathrm{d}u_s \tag{B7}$$

The problem of the differentiation can be overcome by considering the following change of variables. A new position $\hat{S}$ on the line u_S corresponding to the orientation of the moment m_S is defined by

$$\hat{x}_S = x_S + e \cos \delta_S \qquad \text{and} \qquad \hat{y}_S = y_s + e \sin \delta_S. \tag{B8}$$

The velocity $v_{\hat{S}}$ at $(\hat{x}_S, \hat{y}_S)$ is given by

$$v_{\hat{S}} = f_P \sum_{n=1}^{\infty} (\eta\omega\omega_n^2 + \mathrm{j}\omega(\omega_n^2 - \omega^2)) \frac{\varphi_n(\hat{x}_S, \hat{y}_S)\varphi_n(x_P, y_P)}{\Lambda_n[(\omega_n^2 - \omega^2)^2 + \omega_n^4 \eta^2]}. \tag{B9}$$

Using equation (B9) and the foregoing change of variables defined by equations (B8) one can deduce that the angular velocity w_S at (x_S, y_S) is

$$w_S = f_P \sum_{n=1}^{\infty} (\eta\omega\omega_n^2 + \mathrm{j}\omega(\omega_n^2 - \omega^2)) \frac{\dfrac{\mathrm{d}\varphi_n\,(x_S + e\cos\delta_S, y_S + e\sin\delta_S)}{\mathrm{d}e}\,\varphi_n\,(s_P, y_P)}{\Lambda_n\,[(\omega_n^2 - \omega^2)^2 + \omega_n^4\,\eta^2]}. \quad \text{(B10)}$$

With $\varphi_n(x_S + e\cos\delta_S, y_S + e\sin\delta_S) = \sin(n_1\,\pi(x_S + e\cos\delta_S)/l_1)\sin(n_2\,\pi(y_S + e\sin\delta_S)/l_2)$ and after some manipulations one finds that w_S is at (x_S, y_S), obtained by letting $e \to 0$, is

$$w_S = f_P \sum_{n=1}^{\infty} (\eta\omega\omega_n^2 + \mathrm{j}\omega(\omega_n^2 - \omega^2)) \frac{\psi_n^{\delta_S}\,(x_S, y_S)\varphi_n\,(x_P, y_P)}{\Lambda_n\,[(\omega_n^2 - \omega^2)^2 + \omega_n^4\,\eta^2]}, \quad \text{(B11)}$$

where $\psi_n^{\delta_S}$ is given by equation (B4). As the angular velocity induced at (x_S, y_S) by a primary force f_P located at (x_P, y_P) is given by $w_S = M'_{SP} f_P$, one thus has

$$M'_{SP} = \sum_{n=1}^{\infty} (\eta\omega\omega_n^2 + \mathrm{j}\omega(\omega_n^2 - \omega^2)) \frac{\psi_n^{\delta_S}\,(x_S, y_S)\varphi_n\,(x_P, y_P)}{\Lambda_n\,[(\omega_n^2 - \omega^2)^2 + \omega_n^4\,\eta^2]}, \quad \text{(B12)}$$

which is exactly, as expected, the same expression as in equation (B5). By using the same procedure the mobility term that for the angular velocity w_P induced at (x_P, y_P) by a moment m_S located at (x_S, y_S) can be determined:

$$M'_{PS} = \sum_{n=1}^{\infty} (\eta\omega\omega_n^2 + \mathrm{j}\omega(\omega_n^2 - \omega^2)) \frac{\psi_n^{\delta_P}\,(x_P, y_P)\varphi_n\,(x_S, y_S)}{\Lambda_n\,[(\omega_n^2 - \omega^2)^2 + \omega_n^4\,\eta^2]}. \quad \text{(B13)}$$

It can be noted that $M'_{PS} \neq M'_{SP}$. The analytic expression for M'_{PP} or M'_{SS} is exactly the same as, respectively, in equation (B12) or (B13). The mobility M'_{PP} will refer to $\varphi_n\,(x_P, y_P)$ and $\psi_n^{\delta_P}\,(x_P, y_P)$, while the mobility M'_{SS} will refer to $\varphi_n\,(x_S, y_S)$ and $\psi_n^{\delta_S}\,(x_S, y_S)$.

The formulation presented here for M'_{SP} and M'_{PS} is valid for any pair of positions on which act moments and forces that interfere.

B.3 ANALYTIC EXPRESSIONS FOR P_{SS}, P_{PP}, P_{PS} AND P_{SP}

One wants to find the analytic expression for the *angular* mobility P_{PS} in the case of primary and secondary moments, relevant to, for example, the angular velocity w_P induced at (x_P, y_P) by the secondary moment m_P located at (x_S, y_S). The linear velocity induced by a moment is given by equation (B3). Using equation (B6) and considering the differentiation at (x_P, y_P), one has immediately the expression for w_P,

$$w_P = m_S \sum_{n=1}^{\infty} (\eta\omega\omega_n^2 + \mathrm{j}\omega(\omega_n^2 - \omega^2)) \frac{\psi_n^{\delta_P}\,(x_P, y_P)\psi_n^{\delta_S}\,(x_S, y_S)}{\Lambda_n\,[(\omega_n^2 - \omega^2)^2 + \omega_n^4\,\eta^2]}, \quad \text{(B14)}$$

where $\psi_n^{\delta_P}$ is given by equation (B4) by using the orientation of the primary moment δ_P and the co-ordinates (x_P, y_P).

The angular velocity w_P at (x_P, y_P) induced by a secondary moment m_S located at (x_S, y_S) is given by $w_P = P_{PS}\, m_S$. From equation (B14) one deduces the analytic expression for the *angular* mobility P_{PS} :

$$P_{PS} = \sum_{n=1}^{\infty} (\eta\omega\omega_n^2 + \mathrm{j}\omega(\omega_n^2 - \omega^2)) \frac{\psi_n^{\delta_P}\,(x_P, y_P)\psi_n^{\delta_S}\,(x_S, y_S)}{\Lambda_n\,[(\omega_n^2 - \omega^2)^2 + \omega_n^4\,\eta^2]}. \quad \text{(B15)}$$

The same procedure can be adopted to find the mobility term for the angular velocity w_S induced at (x_S, y_S) by a moment m_P located at (x_P, y_P); it is

$$P_{SP} = \sum_{n=1}^{\infty} (\eta\omega\omega_n^2 + \mathrm{j}\omega(\omega_n^2 - \omega^2)) \frac{\psi_n^{\delta_S}(x_S, y_S)\psi_n^{\delta_P}(x_P, y_P)}{\Lambda_n[(\omega_n^2 - \omega^2)^2 + \omega_n^4\eta^2]}. \tag{B16}$$

It can be noted that in this case $P_{SP} = P_{PS}$. The analytic expression for P_{SS} or P_{PP} is exactly the same as, respectively, in equations (B15) or (B16). The mobility P_{SS} will refer to $\psi_n^{\delta_S}(x_S, y_S)^2$, while the mobility P_{PP} will refer to $\psi_n^{\delta_P}(x_P, y_P)^2$. It can be that $P_{SP} = P_{PS}$ but $P_{PP} \neq P_{SS}$.

Also, in this case, the formulation presented for P_{PS} and P_{SP} is valid for any pair of positions on which act moments and forces that interfere.